Modelling the yield curve using orthonormalised Laguerre polynomials: An inter-temporally consistent risk-neutral approach, and an economic interpretation

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Abstract

Krippner (2002) introduces the orthonormalised Laguerre polynomial (OLP) model of the yield curve that generalises the popular exponential-polynomial approach originally due to Nelson and Siegel (1987). This article extends the OLP model into a risk-neutral no-arbitrage setting using the Heath, Jarrow and Morton (1992) framework. The resulting volatility-adjusted OLP (VAO) model is closely related to the generic general equilibrium approach to modeling the yield curve proposed by Berardi and Esposito (1999), and this implies an economic interpretation of the

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VAO model parameters. Like the OLP model, the VAO model is parsimonious, intuitive, and straightforward to apply, and combined with its robust no-arbitrage and equilibrium foundations, it is an appealing choice for a wide variety of practical yield curve-related applications.

JEL: E43, C21, C22

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1 Introduction

Nelson and Siegel (1987) proposed the original exponential-polynomial model of the yield curve, and this approach has subsequently been extended and revisited in Svensson (1994), Hunt (1995), Bliss (1997), Mansi and Phillips (2001), and Diebold and Li (2002). Models of this class are widely used by yield curve practitioners,[1] and perform very favourably in comparison with other yield curve models.[2]

Krippner (2002) introduces the orthonormalised Laguerre polynomial (OLP) model of the yield curve that generalises the exponential-polynomial approach to an arbitrary number of orthonormal factors. The OLP model also allows for the simultaneous modelling of other same-currency yield curves that have group-specific differences (such as default risk), as in Houweling, Hoek and Kleibergen (2001). The OLP model is both cross-sectionally consistent (i.e it reliably fits today's yield curve), and inter-temporally consistent within the expectations hypothesis framework (i.e the cross-sectional OLP model leads to an analytical yield curve evolution that remains in precise OLP form).

Notwithstanding their popularity, exponential-polynomial models and the generic OLP model still have two theoretical short-comings. Firstly, models of this class are not inter-temporally consistent within the risk-neutral framework,

[1] Bank for International Settlements (1999) notes that ten central banks (of twelve surveyed) routinely use either the Nelson and Siegel (1987) and/or the Svensson (1994) model as their primary method for analysing the yield curve. Kacala (1993), Barrett, Gosnell and Heuson (1995), Schich (1997), and Söderlind and Svensson (1997), Soto (2001), and Schmid and Kalemanova (2002) are further examples of the practical application of exponential-polynomial models.

[2] See, for example, Dahlquist and Svensson (1996), Seppala and Viertio (1996), Bliss (1997), and Fergusson and Raymar (1998).

as noted in Björk and Christensen (1999), Filopović (1999a), and Filopović (1999b). Such consistency is required in applications where interest rate volatility is an essential consideration (e.g pricing interest rate options). Secondly, the OLP model lacks a fundamental economic foundation. That is, there has been no formal attempt in the literature to relate the levels and changes of exponential-polynomial model parameters back to the underlying economic and financial state variables that should drive the yield curve, as is the basis for equilibrium models of the yield curve.

The purpose of this article is to address the two points raised in the previous paragraph. Regarding inter-temporal consistency within the risk-neutral framework, section 2 applies the Heath, Jarrow and Morton (1992) framework to OLP functions to construct the volatility-adjusted OLP (VAO) model of the forward rate curve. Section 3 shows that the VAO model still retains the precise analytical inter-temporal consistency obtained with the OLP model, which is convenient for analytical and empirical applications. Regarding the link to underlying state variables, section 4 shows that the VAO model is a natural approximation to the generic risk-neutral Vasicek-type multifactor general equilibrium model proposed by Berardi and Esposito (1999), and the economic interpretation of both the OLP and VAO models are discussed in light of this result. Section 5 concludes, and notes several areas for ongoing work.

2 Creating a risk-neutral model of the forward rate curve using orthonormalised Laguerre polynomials

This section derives the volatility-adjusted OLP (VAO) model of the forward rate curve by applying the Heath, Jarrow and Morton (1992) (hereafter HJM) framework to a representation based on OLP functions. Specifically, section 2.1 defines the HJM framework and terminology relevant to this article, section 2.2 proposes an OLP representation for the expected path of the short rate, section 2.3 proposes an OLP representation for forward rate volatility, and section 2.4 draws these elements together to construct the VAO model of the forward rate curve. Section 2.5 makes several observations about the VAO model using prior results from related literature.

2.1 Notation and summary of the HJM framework

The original HJM article specifies the following differential equation for the forward rate process:[3]

$$df\,(x,T) = \alpha\,(x,T)\;dt + \sum_{i=1}^{I} \sigma_i\,(x,T)\;d\tilde{W}_i\,(x) \tag{1}$$

where:

- $f\,(x,T)$ is the forward rate for time T, measured at time x $(T \geq x)$;

[3] Baxter and Rennie (1996) pages 128 to 177, and de La Grandville (2001) pages 359 to 381 contain very accessible expositions of the HJM framework.

- $\alpha(x,T)$ is the drift of the forward rate process, as specified in equation 2 below;
- $\sigma_i(x,T)$ is the volatility function for the forward rate process i;
- $\tilde{W}_i(x)$ is an independent Brownian motion for the forward rate process i (so $d\tilde{W}_i(x)$ are independent Wiener variables); and
- I is is the number of independent factors for the forward rate process.

According to the results of HJM, $\alpha(x,T)$ is naturally constrained to be a function of the forward rate volatility processes, i.e:

$$\alpha(x,T) = \sum_{i=1}^{I} \sigma_i(x,T) \int_x^T \sigma_i(x,u)\, du \tag{2}$$

The solution to equation 1 is:

$$f(x,T) = f(0,T) + \int_0^x \alpha(s,T)\, ds + \sum_{i=1}^{I} \int_0^x \sigma_i(s,T)\, d\tilde{W}_i(s) \tag{3}$$

where $f(0,T)$ is the forward rate for time T, measured at time $x=0$.

The process for the instantaneous short rate, $r(x)$ where $0 \leq x < \infty$, may be obtained by taking the limit as $T \to x$ (since $\lim_{T\to x} f(x,T) = f(x,x) = r(x)$), i.e:

$$r(x) = f(0,x) + \lim_{T\to x} \int_0^x \alpha(s,T)\, ds + \sum_{i=1}^{I} \int_0^x \sigma_i(s,x)\, d\tilde{W}_i(s) \tag{4}$$

Hence, at a given point in time (which defines $x=0$), the HJM framework requires a strict relationship between the expected path of the instantaneous short rate, the current observed/measured forward rate curve, and the volatility functions that effect instantaneous stochastic changes to the forward rate curve as time evolves from the given point in time.

From this point on, a specific time and maturity notation is adopted to avoid any potential ambiguity between observations at a given point in time, and expectations from that given point in time. Hence, in equation 4 maturity m is substituted for x and the result is specifically denoted as an expectation at time t, i.e:

$$E_t\left[r\left(m\right)\right] = f(t, m) + \lim_{T\to m}\int_0^m \alpha\left(s, T\right) ds \tag{5}$$

where:

- $E_t\left[r\left(m\right)\right]$ represents, as at time t, the expected path of the short rate as a function of maturity m $(0 \leq m < \infty)$;
- $f\left(t, m\right)$ represents, as at time t, the forward rate curve as a function of maturity m; and
- $\lim_{T\to m}\int_0^m \alpha\left(s, T\right) ds$ is a time-invariant function of maturity m, and so a time notation is not required.[4]

2.2 An OLP representation for the expected short rate

As detailed in Krippner (2002), the OLP representation for the base curve is simply a linear combination of a constant and the specified number of orthonor-

[4] If the drift term was time-varying, i.e deterministic and/or stochastic, then this could be written as $\left[\lim_{T\to m}\int_0^m \alpha\left(s, T\right) ds\right]_t$ to denote the time-varying aspect.

malised Laguerre polynomials with argument $2\phi m$, i.e $\varphi_p(2\phi m) = \exp(-\phi m) \cdot L_p(2\phi m)$; $L_p(2\phi m) = \sum_{k=0}^{p} \frac{(-1)^k p!(2\phi m)^k}{(k!)^2 (p-k)!}$.[5] A "dummy" spread function allows for the simultaneous representation of curves relative to the base curve (to allow for group-specific differences, such as credit rating).[6] In general then, define $E_t[r(m)]$ as an OLP(N, L) representation, i.e:

$$
\begin{aligned}
E_t[r(m)] &= \sum_{n=1}^{N+L} \beta_n(t) \cdot g_n(\phi, m) && \text{(6a)} \\
&= \left[\boldsymbol{\beta}_{N+L}(t)\right]' \cdot \mathbf{g}(\phi, m) && \text{(6b)}
\end{aligned}
$$

where:

- N is the number of linear parameters in the OLP model for the base curve;
- L is the number of spread functions relative to the base curve;
- $\beta_n(t)$ are the linear coefficients associated with the functions $g_n(\phi, m)$;
- $g_n(\phi, m)$ (also referred to as "modes") are functions of maturity m, i.e $g_1(\phi, m) = 1$, $g_n(\phi, m) = -\varphi_{n-2}(2\phi m)$ for $n > 1$, and $g_{N+l}(\phi, m) = 1 - \exp(-\phi m) = g_1(\phi, m) + g_2(\phi, m)$;
- ϕ is a fixed positive constant;

[5] See, for example, Courant and Hilbert (1953) pages 93 to 97, and Rainville and Bedient (1981) pages 395 to 396.

[6] The instruments must be of a relatively high credit rating for the zero-bounded and monotonically-increasing $g_{N+l}(\phi, m)$ function given here to apply. See, for example, Jarrow, Lando and Turnbull (1997), and Helwege and Turner (1999).

- $\boldsymbol{\beta}_{N+L}(t)$ is the $(N+L)$-vector of coefficients β_n; and

- $\mathbf{g}(\phi, m)$ is the $(N+L)$-vector function containing the values of the modes $g_n(\phi, m)$.

In practice, L is set according to the number of different classes of instruments being modelled, and ϕ is calibrated with regard to historical data. N is determined empirically from historical data depending on the parsimony versus the goodness of fit required by the user. Prior empirical work in Krippner (2002) indicates that the OLP$(3,1)$ model with $\phi = 1$ is sufficient to model New Zealand government-risk and bank-risk interest rate data simultaneously,[7] and so the $N = 3/L = 1$ case is used as an example in this article. The first three short rate modes for the government-risk curve (called the Level, Slope, and Bow modes) and the short rate spread function for the bank-risk curve (called the Spread mode) are defined in equations 7a to 7d, and illustrated in figure 1.

$$g_1(\phi, m) = 1 \tag{7a}$$

$$g_2(\phi, m) = -\exp(-\phi m) \tag{7b}$$

$$g_3(\phi, m) = -\exp(-\phi m)(-2\phi m + 1) \tag{7c}$$

$$g_{3+1}(\phi, m) = \begin{cases} 0 & \text{if cashflow is govt-risk} \\ g_1(\phi, m) + g_2(\phi, m) & \text{if cashflow is bank-risk} \end{cases} \tag{7d}$$

[Figure 1 here]

[7] Nelson and Siegel (1987), Barrett et al. (1995), and Diebold and Li (2002) use the equivalent of the OLP$(3,0)$ model (i.e just a base curve) for United States government-risk interest rate data, with ϕ fixed at 7.30, 0.33, and 1.37, respectively.

For later use, the boundary values of $g_n(\phi, m)$ for $n > 1$ are all $g_n(\phi, 0) = -1$, and $\lim_{m\to\infty} g_n(\phi, m) = 0$. The boundary values of $g_{N+l}(\phi, m)$ are all $g_{N+l}(\phi, 0) = 0$, and $\lim_{m\to\infty} g_{N+l}(\phi, 0) = 1$ if the cashflow is bank-risk and zero otherwise.

2.3 An OLP representation for forward rate volatility

If $E_t [r(m)]$ is defined by the OLP(N, L) representation at all points in time, then instantaneous stochastic changes to $E_t [r(m)]$ will also be defined by the same OLP(N, L) modes (i.e stochastic changes will be reflected by unanticipated changes to the coefficient vector $\boldsymbol{\beta}_{N+L}(t)$). Further, equations 3 and 4 imply that instantaneous stochastic changes to $E_t [r(m)]$ are identical to those for $f(t, m)$. Hence, the volatility functions for the forward rate curve are defined by the same OLP(N, L) modes used to represent the short rate, i.e:

$$\sigma(m) = \sum_{i=1}^{N+L} \sigma_n \cdot g_n(\phi, m) \tag{8a}$$

$$= \boldsymbol{\sigma}'_{N+L}\mathbf{g}(\phi, m) \tag{8b}$$

where:

- σ_n (> 0) is the constant volatility coefficient for the function $g_n(\phi, m)$; and
- $\boldsymbol{\sigma}_{N+L}$ is the constant $(N + L)$-vector of volatility coefficients σ_n.

Note that $\boldsymbol{\sigma}'_{N+L}\mathbf{g}(\phi, m)$ is time-invariant, since $\boldsymbol{\sigma}'_{N+L}$ is constant and $\mathbf{g}(\phi, m)$

is time-invariant. Also note that by using the OLP formulation with three modes in the base curve, equation 8 incorporates the "humped" forward rate volatility curves that are typically observed in practice.[8]

Having defined the volatility functions for the forward rate curve, the drift term in the HJM framework is also now defined. For notional convenience, denote the drift term by the series of its underlying components, i.e:

$$\lim_{T \to m} \int_0^m \alpha(s,T)\, ds = \sum_{n=1}^{N+L} \sigma_n^2 \cdot h_n(\phi, m) \tag{9}$$

$$= \mathbf{v}'_{N+L}\mathbf{h}(\phi, m) \tag{10}$$

where:

- $h_n(\phi, m)$ represents the effect that a unit of variance in mode $g_n(\phi, m)$ has on the shape of the forward rate curve;
- $\mathbf{v}_{N+L}$ is the constant $(N+L)$-vector of variance coefficients σ_n^2; and
- $\mathbf{h}(\phi, m)$ is the $(N+L)$-vector function containing the values of the functions $h_n(\phi, m)$.

Note that $\mathbf{v}'_{N+L}\mathbf{h}(\phi, m)$ is time-invariant, since $\mathbf{v}_{N+L}$ is constant and $\mathbf{h}(\phi, m)$ is time-invariant.

It now remains to calculate the functional form for each of the components $h_n(\phi, m)$. The result for $n = 1$, as noted in equation 12a, is already well known.[9]

[8] See Hull (2000) pages 541 to 542 and pages 602 and 603 for further discussion on this.
[9] See, for example, Heath et al. (1992) pages 90 to 91.

For $n > 1$, Appendix A shows that:

$$h_n(\phi, m) = \frac{1}{2\phi^2} \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \cdot (k! - \Gamma[1+k, \phi m])^2 \tag{11}$$

The results for $n = 2$ and $n = 3$ are given in equations 12b and 12c,[10] and the result for the Spread mode is obtained directly by adding the $n = 1$ and $n = 2$ results together. The $h_n(\phi, m)$ functions for $n = 1$ to 3 are illustrated in figure 2.

$$\begin{aligned}
h_1(\phi, m) &= \frac{1}{2} m^2 && \text{(12a)} \\
h_2(\phi, m) &= \frac{1}{2\phi^2} [1 - \exp(-\phi m)]^2 && \text{(12b)} \\
h_3(\phi, m) &= \frac{1}{2\phi^2} [1 - \exp(-\phi m)]^2 \\
&\quad - \frac{1}{\phi^2} [1 - \exp(-\phi m) - \phi m \exp(-\phi m)]^2 && \text{(12c)} \\
h_{3+1}(\phi, m) &= \begin{cases} 0 & \text{if cashflow is govt-risk} \\ h_1(\phi, m) + h_2(\phi, m) & \text{if cashflow is bank-risk} \end{cases} && \text{(12d)}
\end{aligned}$$

[Figure 2 here]

For later use, the boundary values of $h_n(\phi, m)$ for $n > 1$ are $h_n(\phi, 0) = 0$ and $\lim_{m \to \infty} h_n(\phi, m) = \frac{1}{2\phi^2} \cdot \sum_{k=0}^{p} \frac{(-2)^k p!}{(p-k)!}$, as shown in Appendix A. For $n = 1$ and $n = 3 + 1$, $h_n(\phi, 0) = 0$ and $\lim_{m \to \infty} h_n(\phi, m)$ is undefined due to the presence of the quadratic term in maturity. This may seem problematic in principle,

[10] The $n = 2$ result is analogous to the example in Heath et al. (1992) pages 91 to 92 using the volatility function $\exp(-\lambda m/2)$. de La Grandville (2001) pages 368 to 372 also contains an example identical to equation 12b.

since it implies unbounded forward rates with increasing maturity, but section 4.3 discusses why this is not necessarily a cause for concern in practice.

2.4 The volatility-adjusted OLP model of the forward rate curve

Substituting the definition for $E_t\left[r\left(m\right)\right]$ from section 2.2 and the results from section 2.3 into equation 5 gives:

$$\left[\boldsymbol{\beta}_{N+L}\left(t\right)\right]'\mathbf{g}\left(\phi,m\right)=f(t,m)+\mathbf{v}'_{N+L}\mathbf{h}(\phi,m) \tag{13}$$

To be consistent with equation 5, the forward rate curve at time t must be defined as:

$$f(t,m)=\left[\boldsymbol{\beta}_{N+L}\left(t\right)\right]'\mathbf{g}\left(\phi,m\right)-\mathbf{v}'_{N+L}\mathbf{h}(\phi,m) \tag{14}$$

Equation 14 is the generic volatility-adjusted OLP model of the forward rate curve, or the VAO model. Substituting the functions in equations 7 and 12 specifies the VAO$(3,1)$ model. If a representation with more than three base modes is required by the user, then the outline in section 2.2 may be used to specify the additional modes $g_n(\phi,m)$, and equation 11 may be used to specify the related $h_n(\phi,m)$ functions. All additional Spread modes $h_{N+l}(\phi,m)$ have the same form as equation 12d.

For later comparison, the boundary values of $E_t\left[r\left(m\right)\right]$ are $E_t\left[r\left(0\right)\right]=\beta_1-\sum_{n=2}^{N}\beta_n$ and $\lim_{m\to\infty}r(t,m)=\beta_1$. The lower boundary value for $f(t,m)$ is $f(t,0)=\beta_1-\sum_{n=2}^{N}\beta_n$ and the upper boundary value will be discussed in section 4.3.

2.5 Discussion of the proposed VAO model

The first point to note is that the VAO model incorporates the OLP model; i.e the VAO model will "collapse" to the OLP model in the purely deterministic case (i.e the complete absence of volatility/uncertainty), and the pure expectations hypothesis will apply precisely. That is, if $\mathbf{v} = \mathbf{0}$, then $f(t,m) = [\boldsymbol{\beta}(t)]' \mathbf{g}(\phi, m) = E_t[r(m)]$.

Secondly, it is apparent from the form of the VAO model functions that the OLP model of the forward rate curve cannot be inter-temporally consistent within the risk-neutral setting, except for the purely deterministic case. That is, if the initial forward rate curve and instantaneous stochastic changes to that forward rate curve are both represented by OLP functions of the type $\exp(-\phi m) \cdot L_n(2\phi m)$, then the HJM framework specifies that the future forward rate curve will include OLP functions of the type $\exp(-2\phi m) \cdot L_n(4\phi m)$. Since the latter cannot be expressed precisely as linear combinations of $\exp(-\phi m) \cdot L_n(2\phi m)$, then the OLP model cannot, as time evolves, continue to precisely represent the forward rate curve in a way consistent with the initial OLP model of the forward rate curve. This is the essence of the results in Björk and Christensen (1999), Filopović (1999*a*), and Filopović (1999*b*), which is why Filopović (1999*a*) argues against the use of bounded exponential-polynomial functions (such as the OLP model) to model the yield curve.

Thirdly, the VAO model may be seen as a "manifold expansion" of the OLP model, analogous to that suggested by Björk and Christensen (1999) (pages 338 to 339) to make the (bounded exponential-polynomial) Nelson and Siegel (1987) model consistent with the risk-neutral no-arbitrage Hull and White (1990) model. That is, adding functions of the form $\exp(-2\phi m) \cdot L_n(4\phi m)$ is required to make the original OLP model inter-temporally consistent within a risk-neutral setting.[11] The m^2 term in the VAO model is a manifold expansion to account for the effect of volatility in the Level mode, and this term also occurs in other risk-neutral models that use a constant forward rate volatility for all maturities.[12]

Fourthly, the VAO model is of the no-arbitrage class in the sense noted by Brandt and Yaron (2002). That is, if a precise fit to market-observed data is required, then the number of modes may in principle be increased to equal the number of instruments. However, in practice and as noted by Brandt and Yaron (2002), the user will often prefer a more parsimonious representation (i.e an approximately arbitrage-free model) in the knowledge that market-observed data contains "measurement errors".[13] Indeed, an advantage of the VAO model is that the generic specification enables the user to choose the precision versus parsimony compromise that best suits their particular application.

Finally, the routine empirical application of the VAO model to cross-sections of yield curve data is as straightforward as the OLP model. That is, once ϕ and the variance coefficients $\mathbf{v}'_{N+L}$ are calibrated from historical data (or

[11] Diebold and Li (2002) also notes this as a potential extension of the exponential-polynomial approach to modelling the yield curve.

[12] For example, the Vasicek (1977) model with zero mean-reversion (as noted in Hull 2000, page 567), the Ho and Lee (1986) model (as noted in Hull 2000, pages 108 and 572 to 574), and the Heath et al. (1992) constant volatility model (pages 90 to 91).

[13] For example, the last-traded price may be out of date. Even a live quoted mid-price is an approximation to the "true" market price; the latter will lie somewhere between the bid and ask quotes, and so the measurement precision is ultimately limited by the smallest allowed or conventional quantum in the market quotes.

perhaps from current interest rate options data in the case of $\mathbf{v}'_{N+L}$), then the estimation of the VAO model is simply an optimisation excercise to obtain the linear coefficients $\boldsymbol{\beta}_{N+L}(t)$.[14]

3 The time evolution of the VAO$(3,1)$ parameters

To provide a convenient basis for analytical and empirical work, this section explicitly illustrates the inter-temporal consistency of the VAO model. Specifically, section 3.1 defines the expected evolution of the short rate, and section 3.2 applies this definition to the VAO$(3,1)$ model as an example. Analogous results for other VAO(N,L) models may be derived using the procedure outlined below.

3.1 The expected path of the short rate

The initial shape of $E_t[r(m)]$ at a point in time implies an expected evolution of $E_t[r(m)]$ over time. This may be expressed as:

$$E_t\{E_{t+\tau}[r(m)]\} = a(m) + E_t[r(m+\tau)] \tag{15}$$

where:

- E_t is the expectation operator conditional upon information available at time t;
- τ is a positive increment of time, measured in years;

[14] The calculation of the interest rate functions associated with $\mathbf{g}(\phi,m)$ and $\mathbf{h}(\phi,m)$, and the empirical application of the VAO model will be included in future work by the author.

- $E_{t+\tau}[r(m)]$ is the expected path of the short rate at time $t+\tau$;

- $E_t[r(m+\tau)]$ is the expected path of the short rate, measured at time t; and

- $a(m)$ allows for a term premium, which is assumed to be a time-invariant function of maturity.

3.2 The expected path of the short rate using the OLP(3, 1) representation

As specified in section 2.2, both $E_t\{E_{t+\tau}[r(m)]\}$ and $E_t[r(m+\tau)]$ for the VAO(3, 1) model will be defined by the OLP(3, 1) representation, i.e $E_t[\boldsymbol{\beta}(t+\tau)]'\,\mathbf{g}(\phi,m)$ and $E_t[\boldsymbol{\beta}(t)]'\,\mathbf{g}(\phi,m+\tau)$, respectively. Substituting these into equation 15 gives the equality:

$$E_t\left\{[\boldsymbol{\beta}(t+\tau)]'\,\mathbf{g}(\phi,m)\right\} = \boldsymbol{\alpha}'\mathbf{g}(\phi,m) + [\boldsymbol{\beta}(t)]'\,\mathbf{g}(\phi,m+\tau) \qquad (16)$$

where $\boldsymbol{\alpha}'\mathbf{g}(\phi,m) = a(m)$, with $\boldsymbol{\alpha}'$ constant and $\mathbf{g}(\phi,m)$ time-invariant. $\mathbf{g}(\phi,m+\tau)$ may be expressed precisely in terms of $\mathbf{g}(\phi,m)$,[15] which means the expected evolution of the OLP(3, 1) model parameters may be written as:

$$E_t\left\{[\boldsymbol{\beta}(t+\tau)]'\right\}\mathbf{g}(\phi,m) = \boldsymbol{\alpha}'\mathbf{g}(\phi,m) + [\boldsymbol{\beta}(t)]'\,[\boldsymbol{\Phi}(\phi,\tau)]'\,\mathbf{g}(\phi,m) \qquad (17)$$

[15] This is evident by re-expressing each $g_n(\phi,m+\tau)$ in terms of $g_n(\phi,m)$. For example, $g_2(\phi,m+\tau) = -\exp(-\phi[m+\tau]) = \exp(-\phi\tau)\cdot -\exp(-\phi m) = \exp(-\phi\tau)\cdot g_2(\phi,m)$. The other relevant results are: $g_1(\phi,m+\tau) = g_1(\phi,m)$; $g_3(\phi,m+\tau) = -2\phi\tau\exp(-\phi\tau)\cdot g_2(\phi,m) + \exp(-\phi\tau)\cdot g_3(\phi,m)$; and $g_{3+1}(\phi,m+\tau) = [1-\exp(-\phi\tau)]\cdot g_1(\phi,m) + \exp(-\phi\tau)\cdot g_{3+1}(\phi,m)$. Analogous results follow for $N > 3$, if the user requires an extended model for the base forward rate curve.

where:

- $[\mathbf{\Phi}(\phi,\tau)]' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \exp(-\phi\tau) & 0 & 0 \\ 0 & -2\phi\tau\exp(-\phi\tau) & \exp(-\phi\tau) & 0 \\ 1-\exp(-\phi\tau) & 0 & 0 & \exp(-\phi\tau) \end{bmatrix}$.

Factoring out the common term $\mathbf{g}(\phi,m)$, and then taking the transpose gives the final result:

$$E_t\{\boldsymbol{\beta}(t+\tau)\} = \boldsymbol{\alpha} + \mathbf{\Phi}(\phi,\tau)\boldsymbol{\beta}(t) \tag{18}$$

Therefore, the process for the expected evolution of the VAO$(3,1)$ model parameters $\boldsymbol{\beta}(t)$ under the risk-neutral framework is identical to that for the OLP$(3,1)$ model under the expectations hypothesis (although the levels of the estimated parameters $\boldsymbol{\beta}(t)$ will differ between the two models for a given cross-section of yield curve data).

Equations 18 and 14 provide the expected evolution of the forward rate curve under the risk-neutral framework, i.e:

$$\begin{aligned} E_t[f(t+\tau,m)] &= E_t\left\{[\boldsymbol{\alpha}+\boldsymbol{\beta}(t+\tau)]'\right\}\mathbf{g}(\phi,m) - \mathbf{v}'_{N+L}\mathbf{h}(\phi,m) && \text{(19a)} \\ &= [\boldsymbol{\alpha}+\mathbf{\Phi}(\phi,\tau)\boldsymbol{\beta}(t)]'\,\mathbf{g}(\phi,m) - \mathbf{v}'_{N+L}\mathbf{h}(\phi,m) && \text{(19b)} \end{aligned}$$

where, as already noted in section 2.3, $\mathbf{v}'_{N+L}\mathbf{h}(\phi,m)$ is time-invariant. Hence, if a prediction of the forward rate curve using today's fitted forward rate curve is required (e.g to value claims contingent on interest rates or the yield curve at a future point in time), then equation 19b provides a convenient basis for such a prediction.

4 Relating the OLP model to a multifactor equilibrium model of the forward rate curve

To provide a cross-check of the form of the proposed VAO model and to establish an economic interpretation of the VAO model parameters, this section compares the VAO model to a risk-neutral equilibrium forward rate model. Specifically, section 4.1 summarises a generic general equilibrium model of the forward rate curve, section 4.2 shows that the real component of this model is naturally approximated by OLP functions, section 4.3 shows that the nominal component of this model is represented by the Level mode of the VAO model, and section 4.4 draws together these results to show that the VAO model of the forward rate curve is the natural approximation to the generic general equilibrium model of the forward rate curve. Section 4.5 introduces default-risk into the BE-V model, and section 4.5 discusses the economic interpretation of the VAO model parameters.

4.1 A multifactor equilibrium model of the forward rate curve

Berardi and Esposito (1999) derives a generic multifactor affine model of the forward rate curve from a general equilibrium model based on the economic model proposed by Cox, Ingersoll and Ross (1985*a*). The Berardi and Esposito (1999) approach encapsulates all Vasicek-type and Cox-Ingersoll-Ross-type[16] equilibrium models, and many other equilibrium models that have been proposed in the literature. It also encapsulates the affine multifactor models of Duffie and Kan (1996) and Dai and Singleton (2000), providing a general equilibrium basis for those models and explicitly accounting for the separation between real and nominal variables.

The Berardi and Esposito (1999) generic risk-neutral J-factor Vasicek-type process (hereafter denoted the BE-V process) is:

$$ds_j = \kappa_j \left(\theta_j - s_j\right) dt + \sigma_j dz_j \tag{20}$$

where, for $j = 2$ to J:

- s_j are the real state variables, representing returns on factors of production in the economy. These are constructed from the original state variables to be mutually uncorrelated.
- κ_j (> 0) is the constant mean-reversion coefficient of the process for s_j;
- θ_j (> 0) is the constant long-term value of s_j;
- σ_j (> 0) is the constant standard deviation of the stochastic process for s_j; and
- dz_j are independent Wiener variables.

[16] That is, Gaussian and square root dynamics, respectively. See the original article, Vasicek (1977), or Hull (2000) page 567 for a summary of the Vasicek equilibrium model, and the original article, Cox, Ingersoll and Ross (1985*b*), or Hull (2000) page 570 for a summary of the Cox-Ingersoll-Ross equilibrium model.

The $j = 1$ factor is reserved for an inflation state variable, as discussed in section 4.3. The BE-V model of the forward rate curve is derived from the combination of processes in equation 20, and is given in Berardi and Esposito (1999) as:

$$\begin{aligned} f(s_1, \mathbf{s}, m) &= s_1 + \kappa_1 (\theta_1 - s_1) B_1(m) - \frac{1}{2}\sigma_1^2 [B_1(m)]^2 \\ &\quad + \sum_{j=2}^{J} s_j + \sum_{j=2}^{J} \kappa_j (\theta_j - s_j) B_j(m) - \sum_{j=2}^{J} \frac{1}{2}\sigma_j^2 [B_j(m)]^2 \end{aligned} \tag{21}$$

where:

- $f(s_1, \mathbf{s}, m)$ is the current nominal forward rate curve, as a function of the single nominal state variable, s_1, the $(J-1)-$vector of real state variables, $\mathbf{s}$, and maturity m $(0 \leq m < \infty)$. The first line of equation 21 is the inflation component of the nominal forward rate curve, denoted $f(s_1, m)$, and the second line is the real component, denoted $f(\mathbf{s}, m)$.

- $B_j(m) = \frac{1}{\kappa_j}[1 - \exp(-\kappa_j m)]$ (the typical Vasicek-type form) for all j. Note that $B_j(0) = 0$, and $\lim_{m\to\infty} B_j(m) = \frac{1}{\kappa_j}$.

For later comparison, the lower boundary value of the first moment term of $f(\mathbf{s}, m)$, i.e $\sum_{j=2}^{J} s_j + \sum_{j=2}^{J} \kappa_j (\theta_j - s_j) B_j(m)$, is $\sum_{j=2}^{J} s_j$, and the lower boundary value of the second moment term of $f(\mathbf{s}, m)$, i.e $\sum_{j=2}^{J} \frac{1}{2}\sigma_j^2 [B_j(m)]^2$, is zero. The upper boundary values of the first and second moment terms of $f(\mathbf{s}, m)$ are $\sum_{j=2}^{J} \theta_j$ and $\sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2}$, respectively.

4.2 The real components of the BE-V model

The similarity of the real component of equation 21 to the OLP components in the VAO model may be seen by some manipulation of equation 21. Firstly, substituting the specific form of $B_n(m)$ into the real part of equation 21, expanding, and cancelling the summation term $\sum_{j=2}^{J} s_j$ gives:

$$\begin{aligned} f(\mathbf{s}, m) &= \sum_{j=2}^{J} \theta_j - \sum_{j=2}^{J} (\theta_j - s_j) \cdot \exp(-\kappa_j m) - \sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2} \\ &\quad + \sum_{j=2}^{J} \frac{\sigma_j^2}{\kappa_j^2} \cdot \exp(-\kappa_j m) - \sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2} \cdot \exp(-2\kappa_j m) \end{aligned} \tag{22}$$

For $j = 2$ to J, define ϕ as a central measure of the values of κ_j, i.e $\phi = \text{Central}(\kappa_j)$, so that $\kappa_j = \phi[1 + \Delta_j]$ with $-1 < \Delta_j < 1$.[17] Equation 22 may then be written equivalently as:

$$\begin{aligned} f(\mathbf{s}, m) = \sum_{j=2}^{J} \theta_j \quad & - \exp(-\phi m) \cdot \sum_{j=2}^{J} (\theta_j - s_j) \cdot \exp(-\Delta_j \phi m) \\ & - \sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2} + \exp(-\phi m) \cdot \sum_{j=2}^{J} \frac{\sigma_j^2}{\kappa_j^2} \cdot \exp(-\Delta_j m) \\ & - \exp(-2\phi m) \sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2} \cdot \exp(-2\Delta_j m) \end{aligned} \tag{23}$$

Each exponential term containing Δ_j can be approximated by a Taylor expansion around $\Delta_j = 0$. The explicit form for the first exponential summation

[17] This restriction on Δ_j is always possible by construction; in the extreme case, ϕ could be defined as $\max(\kappa_j)$, and then $-1 < \Delta_j \leq 0 < 1$ (since the lower bound for κ_j is zero).

term in equation 23 to order P is:

$$\sum_{j=2}^{J} (\theta_j - s_j) \cdot \exp(-\Delta_j \phi m) \simeq \sum_{j=2}^{J} (\theta_j - s_j) \cdot \left[\sum_{p=0}^{P} \frac{1}{p!} (-\Delta_j \phi m)^p \right] \tag{24}$$

where the residual term in this approximation, $\sum_{p=P+1}^{\infty} \frac{1}{p!} (-\Delta_j \phi m)^p$, will always converge to a finite value since $|\Delta_j| < 1$. A full expansion of the double summation terms in equation 24 and a regrouping by powers of ϕm results in an equivalent expression of the form $\sum_{p=0}^{P} w_p \cdot (\phi m)^p$, where w_p is the resulting summation of coefficients on powers of ϕm.

A linear combination of Laguerre polynomials with argument $2\phi m$ up to order P may be written as:

$$\sum_{p=0}^{I} \alpha_p \cdot L_p(2\phi m) = \sum_{p=0}^{P} \alpha_p \cdot \left[\sum_{k=0}^{p} \frac{(-1)^k \, p! (2\phi m)^k}{(k!)^2 \, (p-k)!} \right] \tag{25}$$

where α_p are the linear coefficients. Once again, a full expansion of the double summation terms in equation 25 and a regrouping by powers of ϕm results in the equivalent expression $\sum_{p=0}^{P} w_p \cdot (\phi m)^p$. Therefore, with suitably chosen coefficients α_p, a linear combination of Laguerre functions $L_p(2\phi m)$ to order P is an equivalent representation of the approximation in equation 24 to order P. A similar argument may be used for the remaining exponential summation terms in equation 23, respectively using linear coefficients χ_p and $L_p(2\phi m)$, and linear coefficients γ_p and $L_p(4\phi m)$.

Finally, using the definition of orthonormalised Laguerre polynomials, $\varphi_p(x) =$

$\exp(-x/2) \cdot L_p(x)$, equation 23 may be written as a Taylor expansion approximation to order P as:

$$
\begin{aligned}
f(\mathbf{s}, m) \simeq & \sum_{j=2}^{J} \theta_j - \sum_{p=0}^{P} \alpha_p \cdot \varphi_p(2\phi m) \\
& - \sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2} - \sum_{p=0}^{P} \chi_p \cdot \varphi_p(2\phi m) - \sum_{p=0}^{P} \gamma_p \cdot \varphi_p(4\phi m) \qquad (26)
\end{aligned}
$$

4.3 The inflation component of the BE-V model

To allow for the nominal side of the economy, Berardi and Esposito (1999) introduces a single independent nominal factor to represent the expected inflation rate. For this factor, each of the parameters in equation 20 are analogous to their real counterparts, although some require a combination of relative price level and expected inflation rate parameters to define them; i.e: $s_1 = \pi - \sigma_{pl}^2$; $\kappa_1 = \kappa_\pi$; $\theta_1 = \theta_\pi - \sigma_{pl}^2$; and $\sigma_1 = \sigma_\pi$, where π is the expected inflation rate, σ_{pl}^2 is the variance of relative changes in the price level, κ_π is the mean-reversion coefficient for the expected inflation rate, θ_π is the long-term expected inflation rate, and σ_π is the standard deviation of the expected inflation rate. Apart from s_1 and π, all of these parameters are positive constants.

The inflation factor has an important analytical difference to the real factors discussed in section 4.2, since the empirical results from Berardi and Esposito (1999) and Brown and Schaefer (1994) indicate that the mean-reversion coefficient for the inflation process, κ_1, is much lower than that for the real variables. Indeed, the point estimates of κ_1 in those articles are distributed above and below zero, and none are statistically different from zero.[18]

The lowest limit of κ_1 is zero, and the inflation component of equation 21 for this particular case may be evaluated by setting $\kappa_1 = 0$ and noting that, by L'Hôpital's rule, $\lim_{\kappa_1 \to 0} B_1(m) = m$, i.e:

$$f(s_1, m) = s_1 - \frac{1}{2}\sigma_1 m^2 \tag{27}$$

Recall that the m^2 term also arose in section 2.6, associated with the Level mode. The analysis above now shows why this offers no cause for concern, since the Level mode may be regarded as a factor with a zero mean-reversion coefficient. Conversely, this suggests that if a VAO model without the m^2 term is required/desired, then the Level mode in sections 2 and 3 could simply be replaced with a low mean-reversion coefficient alternative, i.e $g_1(\lambda, m) = -\exp(-\lambda m)$ where $\lambda \ll \phi$. By reference to equation 12b, $g_1(\lambda, m)$ would give $h_1(\lambda, m) = \frac{1}{2\lambda^2}[1 - \exp(-\lambda m)]^2$, and $\lim_{m \to \infty} h_1(\lambda, m) = \frac{1}{2\lambda^2}$.

4.4 The BE-V model in VAO model form

Combining the real forward rate curve from equation 26 and the inflation component of the forward rate curve from equation 27, the nominal BE-V forward rate curve may be written as a Taylor expansion approximation to order P as:

[18] As noted by Berardi and Esposito (1999), this is consistent with the Fisher hypothesis that changes in nominal long-maturity rates are determined almost exclusively by changes to the expected inflation rate, while changes in nominal short-maturity rates are determined almost exclusively by changes in the real short rate.

$$
\begin{aligned}
f\left(s_1, \mathbf{s}, m\right) \quad \simeq \quad & s_1+\sum_{j=2}^{J} \theta_j-\sum_{p=0}^{P} \alpha_p \cdot \varphi_p(2 \phi m)-\frac{1}{2} \sigma_1 m^2 \\
& -\left[\sum_{j=2}^{J} \frac{\sigma_j^2}{2 \kappa_j^2}+\sum_{p=0}^{P} \chi_p \cdot \varphi_p(2 \phi m)+\sum_{p=0}^{P} \gamma_p \cdot \varphi_p(4 \phi m)\right] \quad (28)
\end{aligned}
$$

Referring to equation 14, equation 28 may be written explicitly in the summation form of the VAO model:

$$
\begin{aligned}
f\left(s_1, \mathbf{s}, m\right) \quad \simeq \quad & \beta_1 \cdot g_1(\phi, m)+\sum_{n=2}^{N} \beta_n \cdot g_n(\phi, m) \\
& -\sigma_1^2 \cdot h_1(\phi, m)-\sum_{n=2}^{N} \sigma_n^2 \cdot h_n(\phi, m) \quad (29)
\end{aligned}
$$

where $g_1(\phi, m)=1$ and $g_n(\phi, m)=-\varphi_{n-2}(2 \phi m)$ for $n>1$ as in section 2.2, and $h_1(\phi, m)=\frac{1}{2} m^2$ and $h_n(\phi, m)$ for $n>1$ are as defined in section 2.3.

4.5 Allowing for default risk in the BE-V model

The BE-V process does not incorporate default risk, and so the BE-V model of the forward rate curve applies only to default-free financial instruments. However, the BE-V model can be adapted to include instruments with default risk by using the technique of Duffie and Singleton (1997). Specifically, Duffie and Singleton (1997) provides the theoretical justification for modelling yield curves that contain default risk using models developed for default-free yield curves, essentially by adding a default factor that is a combination of the probability of default, and the expected loss if default occurs.

A single default factor in the BE-V representation may be denoted as the $J+1$ factor with a state variable s_{J+1}. s_{J+1} is a real state variable, but it differs from the other real state variables because its long-term value θ_{J+1} can no longer be considered a constant; i.e the long-term (or infinite maturity) spread between default-risk and default-free foward rates will change over time according to the market's changing assessment of default-risk. This may be represented as a stochastic process for θ_{J+1} with a zero mean-reversion coefficient. The process of s_{J+1} reverting to θ_{J+1} is allowed for by adding a related stochastic process with a long-term value of zero, i.e:

$$d\theta_{J+1} = \sigma_{J+1} dz_{J+1} \tag{30a}$$

$$ds_{J+1} = -\kappa_{J+1} s_{J+1} dt + \sigma_{J+1} dz_{J+1} \tag{30b}$$

where the stochastic term in both cases, $\sigma_{J+1} dz_{J+1}$, is identical. The combination of these processes will result in precisely the same forward rate curve as equation 21, but with additional first and second moment terms for θ_{J+1} and s_{J+1}, i.e:

$$\begin{aligned} f(s_1, \mathbf{s}, s_{J+1}, m) = {} & f(s_1, \mathbf{s}, m) + \theta_{J+1} - \frac{1}{2}\sigma_{J+1}^2 \left[\lim_{\kappa_{J+1}\to 0} B_{J+1}(m)\right]^2 \\ & + s_{J+1} - \kappa_{J+1} s_{J+1} B_{J+1}(m) - \frac{1}{2}\sigma_{J+1}^2 \left[B_{J+1}(m)\right]^2 \end{aligned} \tag{31}$$

Writing the first second moment term as m^2 (as per section 4.3), and the second first moment term as $-s_{J+1} \cdot \exp\left(-\kappa_{J+1} m\right)$ (as per the first line of equation 22 with $\theta_{J+1} = 0$), results in the expression for the default-risk less default-free forward rates, or the forward default-spread curve, i.e:

$$\begin{aligned} f\left(s_1, \mathbf{s}, s_{J+1}, m\right) - f\left(s_1, \mathbf{s}, m\right) &= \theta_{J+1} - s_{J+1} \cdot \exp\left(-\kappa_{J+1} m\right) \\ &\quad -\sigma_{J+1}^2 \left(\frac{1}{2} m^2 + \frac{1}{2}\left[B_{J+1}\left(m\right)\right]^2\right) \qquad (32) \end{aligned}$$

As noted by Houweling et al. (2001), the forward default-spread curve should equal zero at $m = 0$ and be monotonically increasing by maturity.[19] The first condition requires that $\theta_{J+1} = s_{J+1}$. The second condition is met with any value of κ_{J+1}, although if several forward default-spread curves are being modelled simultaneously then all values of κ_{J+q} (where $q > 1$) must be equal for the difference between these spread curves to be monotonically increasing. The most parsimonious restriction is $\kappa_{J+q} = \phi$, and this also makes the mean-reversion coefficient/s for the default factor/s equal to the central measure of κ_j for the other real factors, since $\phi = \text{Central}(\kappa_j)$. With these two restrictions, the forward default-spread curve, $f\left(s_{J+1}, m\right)$, is defined as:

[19] Assuming a relatively high credit rating; see, Jarrow et al. (1997), and Helwege and Turner (1999).

$$f\left(s_{J+1}, m\right) = s_{J+1} \cdot [1 - \exp(-\phi m)] - \sigma_{J+1}^2 \cdot \left[\begin{array}{c} \frac{1}{2}m^2 + \frac{1}{2\phi^2} - \frac{1}{\phi^2} \cdot \exp(-\phi m) \\ + \frac{1}{2\phi^2} \cdot \exp(-2\phi m) \end{array} \right] \tag{33}$$

This is explicitly of the VAO form contained in section 2, using $g_{3+1}(\phi, m)$ as defined in section 2.2 and $h_{3+1}(\phi, m)$ as defined in section 2.3.

4.6 The economic interpretation of the VAO model parameters, and other observations

In light of the results from the BE-V process above, four observations about the economic interpretation of the VAO model parameters may now be advanced. In addition, since the OLP model differs from the VAO model only by the time-invariant effect that volatility has on the level of the forward rate curve, the observations about changes to the VAO(N, L) linear parameters apply equivalently to changes in the OLP(N, L) linear parameters.

Firstly, the linear coefficients β_n in the base curve of the VAO model reflect the underlying state variables in the economy, s_j; and unanticipated changes to β_n reflect a combination of stochastic changes to s_j. In particular, $\beta_1 = s_1 + \sum_{j=2}^{J} \theta_j$ and so β_1 may be interpreted as the long-run equilibrium nominal interest rate partitioned as the expected inflation rate and the real long-term equilibrium interest rate (a constant). Stochastic changes to β_1 reflect unanticipated changes to s_1, i.e "shocks" to the expected inflation rate.

The second moment terms of the Level mode in the VAO model and the inflation component of the BE-V model are also equal, i.e $-\sigma_1^2 \cdot h_1(\phi, m) = -\frac{1}{2}\sigma_1 m^2$. The levels of the remaining linear coefficients β_n reflect the current values of the underlying real state variables, and there is a precise correspondence at the lower boundary. That is, when $m = 0$, $f(t, 0) - \beta_1 = -\sum_{n=2}^{N} \beta_n$ and $f(\mathbf{s}, 0) = \sum_{j=2}^{J} s_j$, and so $-\sum_{n=2}^{N} \beta_n$ may be interpreted as the current real instantaneous interest rate. Stochastic changes to $\sum_{n=2}^{N} \beta_n$ reflect unanticipated changes to $\sum_{j=2}^{J} s_j$, or "shocks" to the current state of the real economy. Note that the expected path of $-\sum_{n=2}^{N} \beta_n$ at a given point in time can be calculated using the results in section 3.2, and this may be interpreted as expectations of the real instantaneous interest rate and hence the future state of the real economy. For completeness, the upper boundary value for the long-term real forward rate is defined in both the VAO and BE-V models, and in both cases the respective function values at the upper bound increase with the number of modes or factors; i.e $\lim_{m\to\infty} \left[f(t, m) - \beta_1 \cdot g_1(\phi, m) + \sigma_1^2 \cdot h_1(\phi, m)\right] = \sum_{n=2}^{N} \left[\frac{\sigma_n^2}{2\phi^2} \cdot \sum_{k=0}^{n} \frac{(-2)^k (n-2)!}{(n-2-k)!}\right]$, and $\lim_{m\to\infty} f(\mathbf{s}, m) = \sum_{j=2}^{J} \frac{\sigma_j^2}{2\kappa_j^2}$. Therefore, the empirical estimates of the volatility coefficients in the VAO model may be used to obtain an estimate of the second moment term in the BE-V model.[20]

Secondly, the constant parameter ϕ in the base curve of the VAO model may be interpreted as a central measure of the mean-reversion coefficients of the real state variable processes in the BE-V model, i.e $\phi = \text{Central}(\kappa_j)$ (which is also a constant parameter). This correspondence suggests that the VAO model could

[20] But note that $\sum_{k=0}^{n} \frac{(-2)^k (n-2)!}{(n-2-k)!}$ is a divergent function of n, and so a careful analysis of the number of modes and their volatiity would be required in practice.

be used to gauge the expected persistence of real shocks to the economy, e.g with $\phi = 1$ (as in the empirical results of Krippner 2002), 50 percent of a real shock would be expected to dissipate in approximately 0.7 years, and 95 percent in approximately 3 years.

Thirdly, the empirical significance of higher-order modes in the base curve of the VAO model may be seen as an indication of the relative distribution of Δ_j, i.e the magnitudes of the mean-reversion coefficients for the real state variables κ_j relative to Central(κ_j). If the higher-order modes in the VAO model quickly become empirically insignificant, this would represent evidence that the magnitudes of κ_j are generally similar. Conversely, if the empirical significance of higher-order terms remains persistent, this would represent evidence that the distribution of κ_j values is more diffuse, or at least some values of κ_j are materially different to Central(κ_j). The empirical results in Krippner (2002) for New Zealand are consistent with relatively concentrated κ_j values, since an OLP model with only three modes in the base forward rate curve is sufficient to describe the data.

Fourthly, the linear coefficient for the Spread mode in the VAO model reflects the market's current assessment of the long-run credit-risk associated with the instruments that belong to the Spread curve, relative to the default-free yield curve. Specifically, using the VAO$(3, 1)$ model of the government and bank-risk forward curves as an example, $\beta_{3+1} = s_{J+1}$, and so β_{3+1} is the additional yield that investors require to compensate them for the expected loss (i.e the probability of default combined with the expected loss if default occurs) from holding a long-term bank-risk instrument rather than a long-term government-risk instrument. It is also worth noting that if it was thought that a term premium existed in the base forward curve (e.g due to persistent differences between ex-ante expectations versus ex-post realisations of the base forward curve), then this could also be allowed for by incorporating an additional Spread mode in the base curve itself.

As an observation, the Berardi and Esposito (1999) generic risk-neutral J-factor Cox-Ingersoll-Ross-type (CIR) process does not naturally result in exponential-polynomial functions as an approximation to the forward rate curve. Hence, the VAO representation is not a natural approximation to CIR-related processes. That said, it is questionable how applicable the CIR process is in a multifactor model that specifically accounts for the separation of real and nominal state variables. Imposing CIR dynamics on each BE-V process means that none of the state variables can ever assume negative values. In practice, both the inflation rate and the real interest rate have occasionally assumed negative values at times in the past. So, while a CIR process might be desirable in a single-factor model to ensure that nominal short-rates stay positive,[21] this property is not necessarily desirable for each factor in a multifactor model.

As a final observation, one might legitimately ask the question: why use the VAO model instead of just using the BE-V model directly? The first point to note is that the second-order VAO model and the two-factor BE-V model are mathematically equivalent (once all κ_j, θ_j, and σ_j parameters have been calibrated). For higher-order/factor versions, the first reason to prefer the VAO model is empirical: the VAO model uses orthonormal modes which makes for

[21] Duffie (1996) pages 134 to 135 contains a useful discussion on this.

robust estimation, while the BE-V model may be subject to problems associated with multi-collinearity if the mean-reversion coefficients are similar. The VAO model also has less fixed parameters to calibrate than the BE-V model. The second reason is pragmatic: market practitioners intuitively consider the yield curve as a combination of shapes such as the modes in VAO model (e.g for assessing slope risk associated with trading the yield curve); indeed, this is part of the popularity of exponential-polynomial models. Conversely, considering the yield curve as a combination of BE-V factors would not be so readily intuitive.

5 Conclusion

The simplicity of the VAO model combined with its robust no-arbitrage and equilibrium foundations makes it an appealing choice for all practical yield curve-related applications. This includes both purely financial applications (e.g the prediction of the evolution of the yield curve and the pricing of interest rate derivatives), and economic applications (e.g deriving real and nominal economic information from the yield curve, and using the yield curve to imply fundamental economic parameters). Exploring the use of the VAO model in a variety of such applications will be the subject of ongoing work.

A An exponential-polynomial volatility function

Define a time-invariant volatility factor as:

$$\sigma(x,T) = \sigma \exp(-\phi[T-x])[\phi(T-x)]^a \tag{34}$$

where a is an integer ≥ 0. Substituting equation 34 into equation 2 gives:

$$\begin{aligned}
\alpha(x,T) &= \sigma \exp(-\phi[T-x])[\phi(T-x)]^a \\
&\quad \times \int_x^T \sigma \exp(-\phi[u-x])[\phi(u-x)]^a \, du && \text{(35a)} \\
&= \sigma^2 \exp(-\phi[T-x])[\phi(T-x)]^a \\
&\quad \times \left[-\frac{1}{\phi}\Gamma[1+a,\phi(u-x)]\right]_x^T && \text{(35b)} \\
&= \frac{\sigma^2}{\phi}\exp(-\phi[T-x])[\phi(T-x)]^a \\
&\quad \times (-\Gamma[1+a,\phi(T-x)] + \Gamma[1+a,0]) && \text{(35c)}
\end{aligned}$$

where $\Gamma(\cdot)$ is the incomplete Gamma function, and is defined as $\Gamma(a,z) = \int_z^\infty w^{a-1}\exp(-w)\,dw$.[22] This gives the factorial result used for the lower bound of the definite integral since $\Gamma(1+a,0) = \int_0^\infty z^a \exp(-z)\,dz = \Gamma(1+a) = a!$. Evaluating $\int_0^x \alpha(s,x)\,ds$ gives:

[22] See, for example, Wolfram (1996) page 740.

$$
\begin{aligned}
\int_0^x \alpha(s,T)\, ds &= \frac{\sigma^2}{\phi}\int_0^x (\phi[T-s])^a \exp(-\phi[T-s]) \\
&\qquad \times(-\Gamma[1+a,\phi(T-s)]+a!)\, ds && \text{(36a)}\\
&= \frac{\sigma^2}{\phi}\left[\begin{array}{c} \frac{1}{\phi}a!\Gamma[1+a,\phi(T-s)] \\ -\frac{1}{2\phi}(\Gamma[1+a,\phi(T-s)])^2 \end{array}\right]_0^x && \text{(36b)}\\
&= \frac{\sigma_3^2}{2\phi^2}\left[\begin{array}{c} 2a!\Gamma[1+a,\phi(T-s)] \\ -(\Gamma[1+a,\phi(T-s)])^2 \end{array}\right]_0^x && \text{(36c)}\\
&= \frac{\sigma_3^2}{2\phi^2}\left[\begin{array}{c} 2a!\Gamma[1+a,\phi(T-t)] \\ -(\Gamma[1+a,\phi(T-t)])^2 \\ -2a!\Gamma[1+a,\phi T]+(\Gamma[1+a,\phi T])^2 \end{array}\right] && \text{(36d)}
\end{aligned}
$$

$\lim_{T\to x}\int_0^x \alpha(s,T)\, ds$ may be calculated directly by substitution of x for T:

$$
\begin{aligned}
\lim_{T\to x}\int_0^x \alpha(s,T)\, ds &= \frac{\sigma^2}{2\phi^2}\left[\begin{array}{c} 2a!\Gamma[1+a,0]-(\Gamma[1+a,0])^2 \\ -2a!\Gamma[1+a,\phi x]+(\Gamma[1+a,\phi x])^2 \end{array}\right] && \text{(37a)}\\
&= \frac{\sigma^2}{2\phi^2}\left[\begin{array}{c} 2a!\cdot a!-(a!)^2 \\ -2a!\Gamma[1+a,\phi x]+(\Gamma[1+a,\phi x])^2 \end{array}\right] && \text{(37b)}\\
&= \frac{\sigma^2}{2\phi^2}\left[\begin{array}{c} (a!)^2-2a!\Gamma[1+a,\phi x] \\ +(\Gamma[1+a,\phi x])^2 \end{array}\right] && \text{(37c)}\\
&= \frac{\sigma^2}{2\phi^2}(a!-\Gamma[1+a,\phi x])^2 && \text{(37d)}
\end{aligned}
$$

Using the time and maturity notation defined in the main text, the volatility function in equation 34 may be written as $\sigma(m) = \sigma \exp(-\phi m)[\phi m]^a$, and the corresponding result in equation 37d may be written as:

$$\lim_{T\to m} \int_0^m \alpha(s,T)\, ds = \sigma^2 \cdot \frac{1}{2\phi^2} \left(a! - \Gamma\left[1+a, \phi m\right]\right)^2 \tag{38}$$

To calculate $h_n(\phi, m)$ for $n > 1$, the volatility function is written as a summation of exponential-polynomial terms, and the corresponding results using equation 38 are applied. That is, $\sigma_n(m) = \sigma_n \cdot |g_n(\phi, m)| = \sigma_n \cdot \varphi_{n-2}(2\phi m) = \sigma_n \cdot \exp(-\phi m) \cdot \sum_{k=0}^{n-2} \frac{(-1)^k (n-2)!(2\phi m)^k}{(k!)^2 (n-2-k)!} = \sigma_n \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \exp(-\phi m)(\phi m)^k$, and therefore:

$$h_n(\phi, m) = \frac{1}{2\phi^2} \cdot \sum_{k=0}^{n-2} \frac{(-2)^k (n-2)!}{(k!)^2 (n-2-k)!} \cdot \left(k! - \Gamma\left[1+k, \phi m\right]\right)^2 \tag{39}$$

Noting that $\lim_{m\to\infty} \Gamma[1+k, \phi m] = 0$, the upper boundary value of equation 39 is obtained as:

$$\lim_{m\to\infty} h_n(\phi, m) = \frac{1}{2\phi^2} \cdot \sum_{k=0}^{p} \frac{(-2)^k p!}{(p-k)!} \tag{40}$$

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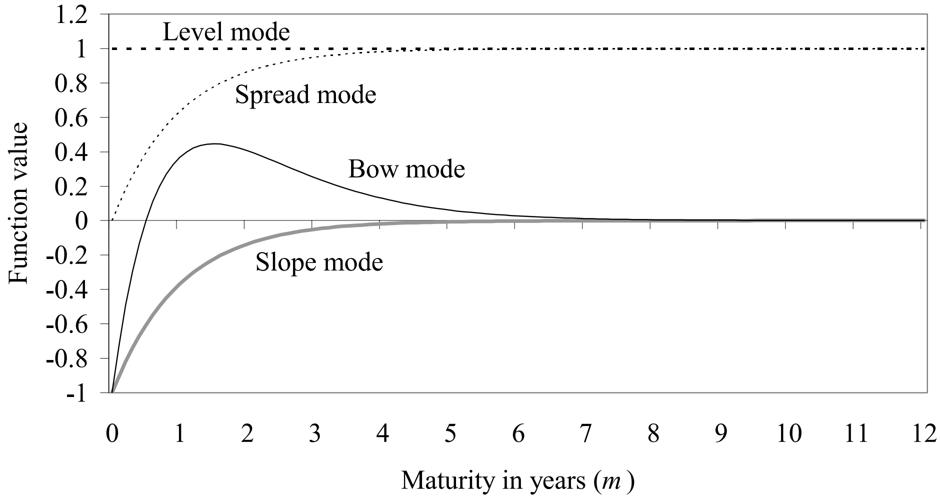

Figure 1: The first three government-risk short rate modes, and the bank-risk short rate Spread mode. Level mode is g_1, Slope mode is g_2, Bow mode is g_3, and Spread mode is g_{3+1}, all with $\phi = 1$.

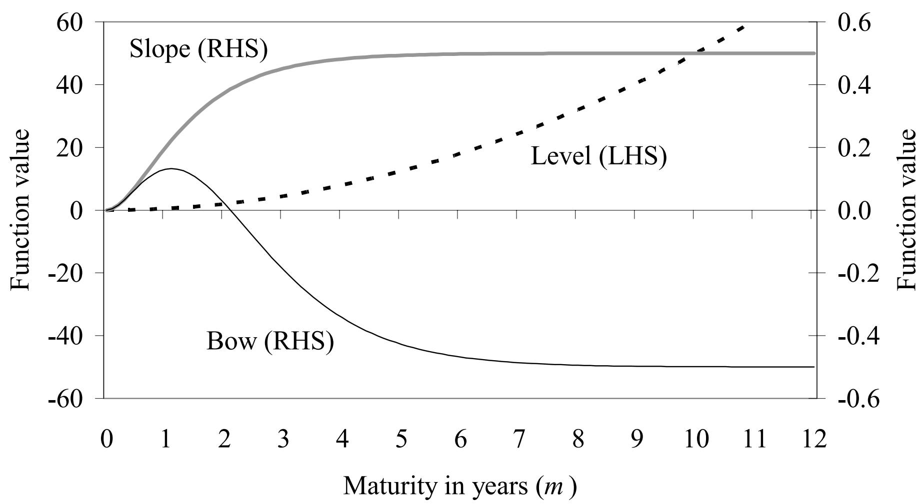

Figure 2: The effect that a unit of variance in mode $g_n(\phi, m)$ will have on the shape of the forward rate curve for the first three short rate modes. Level effect is h_1, Slope effect is h_2, Bow effect is h_3, all with $\phi = 1$. The Spread effect is not shown here because it is almost identical to the Level effect (given the difference in scales of $h_1$and h_2).