Jackknifing Bond Option Prices*

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First version: August 2002
This version: December 2002

*Phillips thanks the NSF for support under Grant No. SES 00-92509. Yu gratefully acknowledges financial support from the Royal Society of New Zealand Marsden Fund under No. 01-UOA-015 and the Cowles Foundation at Yale University for hospitality during a visit over the period from October 2002 to November 2002. We thank Torben Andersen, Federico Bandi, Henk Berkman, Charles Corrado, Tony Hall, Steve Satchell, and seminar participants at Yale University, University of Technology, Sydney for helpful discussions.

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Abstract

In continuous time specifications, the prices of interest rate derivative securities depend crucially on the mean reversion parameter of the associated interest rate diffusion equation. This parameter is well known to be subject to estimation bias when standard methods like maximum likelihood (ML) are used. The estimation bias can be substantial even in very large samples and it translates into a bias in pricing bond options and other derivative securities that is important in practical work. The present paper proposes a very general method of bias reduction for pricing bond options that is based on Quenouille's (1956) jackknife. We show how the method can be applied directly to the options price itself as well as the coefficients in continuous time models. The method is implemented and evaluated here in the Cox, Ingersoll and Ross (1985) model, although it has much wider applicability. A Monte Carlo study shows that the proposed procedure achieves substantial bias reductions in pricing bond options with only mild increases in variance that do not compromise the overall gains in mean squared error.

Our findings indicate that bias correction in estimation of the drift can be more important in pricing bond options than correct specification of the diffusion. Thus, even if ML or approximate ML can be used to estimate more complicated models, it still appears to be of equal or greater importance to correct for the effects on pricing bond options of bias in the estimation of the drift. An empirical application to U.S. interest rates highlights the differences between bond and option prices implied by the jackknife procedure and those implied by the standard approach. These differences are large and suggest that bias reduction in pricing options is important in practical applications.

JEL Classification: C13, C22, E43, G13
Keywords: Bias Reduction, Option Pricing, Bond Pricing, Term Structure of Interest Rate, Re-sampling, Estimation of Continuous Time Models.

1 Introduction

For more than three decades continuous time models have proved to be a versatile and productive tool in finance. Sundaresan (2000) provides a recent extensive survey of these models and their many financial applications. The models are especially useful with respect to pricing derivative securities where both closed form solutions and numerical methods are used in practical work. This paper is concerned with pricing interest rate derivative securities, a practical issue that has been addressed in a number of different ways in the past. One of the oldest and most important approaches is based on modelling the dynamics of the instantaneous interest rate. According to this approach, to calculate prices of derivative securities it is necessary to estimate whatever system parameters occur in the continuous time specification of the underlying asset. Since only discrete time observations are available, a common practice is to discretize the continuous time system and estimate the resulting discretized model (see, for example, Chan, Karolyi, Longstaff, and Sanders (hereafter CKLS), 1992). Unless the exact discrete model is known, as it is in certain special cases (e.g., Phillips, 1972), discretization generally introduces an estimation bias since the internal dynamics between sampling points are ignored. Misspecification bias results in inconsistent estimators (see Merton, 1980, Lo, 1988, and Melino, 1994) of the parameters of the continuous system with consequent bias effects on derivative prices.

To circumvent the problem of inconsistent estimation of continuous systems, methods have been proposed to estimate continuous time specifications directly. Among the techniques that have been proposed, the maximum likelihood (ML) approach is naturally appealing in view of its good asymptotic properties in general regular estimation problems, and maximum likelihood estimation (MLE) has become something of a gold standard to aim for in the estimation of continuous time systems. In consequence, many articles have suggested ways of constructing or approximating the likelihood function of a continuous system analytically and of computing it by numerical or simulation methods. Examples of this approach include Lo (1988), Pedersen (1995), Kessler (1997), Durham and Gallant (2002), Aït-Sahalia (1999, 2002), and Brandt and Santa-Clara (2002), to mention only a few. The ultimate goal of all these methods is to approach the MLE in the hope that this estimator will deliver best performance characteristics.

In spite of its generally good asymptotic properties, the MLE can have a substantial finite sample bias in dynamic models of the type used in financial econometric applications. The bias is well known to be especially acute even in simple models like the first order autoregression (Hurvicz, 1950, Orcutt, 1948) and this bias is exacerbated in models with a fitted intercept and trend (Orcutt and Winokur, 1969, Andrews, 1993, Andrews and Chen, 1994). These bias problems in discrete time dynamic models are manifest in the estimation of continuous time systems, such as diffusion models for short term interest rates (Chapman and Pearson, 2000, Yu and Phillips, 2001) and they persist even when the sample size is quite large, as it often is in financial applications. Similar arguments apply to other commonly used estimation methods in dynamic models, including the general class of extremum estimators.

The problem of estimation bias turns out to be of great importance in the practical use of econometric estimates in asset pricing. The prices of bond options and other derivative securities hinge crucially on the value of unknown parameters. Of particular importance in diffusion models are the parameters governing volatility and drift. When these parameters are estimated with bias, as occurs with the MLE and many other estimation procedures, estimation bias is transmitted to the pricing formulae for bonds, bond options and other derivative securities. For instance, when the true mean reversion parameter is 0.1 and 600 monthly observations are available to estimate a square-root diffusion model (Cox, Ingersoll and Ross, 1985), the bias in the ML estimator of the mean reversion parameter is 84.5% in an upwards direction. This estimation bias further leads to a 24.4% downward bias in the option price of a discount bond and 1.0% downward bias in the discount bond price. The latter figures are comparable in magnitude to the estimates of bias effects discussed in Hull (2000, Chapter 21.7). The biases would be even larger when less observations are available. Of course, these numbers depend on other aspects of the specification, including the nature of the bond and the maturity of the option, which are discussed in Sections 2 and 3. The existence of bias in stock option pricing has been noticed in the literature. For example, it is well known that the Black-Scholes stock option price estimates are biased, even when an unbiased volatility estimate is used (Butler and Schachater, 1986, Knight and Satchell, 1997). However, compared with the documented bias effect in stock option pricing, the bias effect in bond option pricing is found to be much more dramatic in the present paper.

To address the problem of biased estimation in continuous time models with its consequential effects on bond option prices, this paper introduces bias reduction techniques based on the jackknife (Quenouille, 1956). While jackknife methods have been extensively used in discrete time models (e.g., Efron, 1982, and Shao and Tu, 1995), we know of no earlier implementation in continuous time model estimation. The jackknife has several properties that make it appealing in the present application. The first advantage is its generality. Unlike other bias reduction methods, such as those based on corrections obtained by estimating higher order terms in an asymptotic expansion of the bias, the jackknife technique does not rely on the explicit form of an asymptotic expansion. This means that it is applicable in a broad range of model specifications and it is not necessary to develop explicit higher order representations of the bias. In the present context, we can, for instance, apply the jackknife technique directly to the quantity of interest, like the option price itself. Given the complicated form of options price representations in terms of the underlying process and its parameters, this advantage is significant and makes the method very suitable for empirical implementation. In fact, it turns out that direct use of the jackknife to the options price provides significant gains relative to bias reduction in the parameters of the continuous time model. Moreover, other methods of parameter bias reduction in dynamic models, like median unbiased procedures (e.g., Andrews, 1993) are only applicable to parameter estimation and are not directly applicable to more complex quantities like options prices which depend on many other aspects (including distributional details) of the model. A second advantage is that this approach to bias reduction can be used with many different estimation methods, including general methods like MLE. Third, it can be applied in any asset pricing situation (e.g., stock and currency options with stochastic interest rates and interest rate derivatives) where the quantities of interest depend on the estimation of continuous time systems in which finite sample bias arises.

Our findings in this paper indicate that the jackknife provides a very substantial improvement in pricing bond options over existing methods. To illustrate, Fig. 1 compares the distribution of estimates of the option price of a discount bond obtained by using MLE and jackknifed MLE in a Cox, Ingersoll and Ross (CIR) model with 600 monthly observations. As is apparent in the figure, the jackknife estimates are much better centered on the true options price and do not show any appreciable increase in variance. In fact, the root mean squared error (RMSE) of the jackknifed estimates is 12.1% smaller than that of MLE while also providing a bias reduction of 11.5% for 600 observations. Section 3 of the paper explores this implementation of the jackknife in detail and shows that a carefully designed jackknife method can lead to a lower value of RMSE, so bias reduction is accomplished without compromising the gains by much larger variability.

In pricing bond options and interest rate derivatives, model specification is known to be important. For instance, CKLS (1992) show that use of the constant elasticity of variance (CEV) model leads to significant changes in bond option prices compared with alternative models like the CIR or Vasicek model. Concern over specification has also led to the introduction of more flexible methods of estimation, such as the semiparametric treatment of diffusion in Aït-Sahalia (1996a) and the fully nonparametric approaches of Stanton (1997) and Bandi and Phillips (2002) that allow users to be agnostic regarding functional form. At least in models where the drift is linear and parametric, the discrete time equivalent model that is satisfied by equispaced observations has the same general autoregressive form, so that dynamic estimation bias of the type discussed above can be expected in all conventional approaches. In consequence, it may be expected that a bias reduction procedure such as the jackknife may be useful even in situations where the continuous time model is misspecified by incorrect specification of the diffusion. Our findings indicate that the jackknife indeed continues to deliver bias reduction in both autoregressive parameter estimation and in pricing bond options under model misspecification. In fact, the results suggest that bias reduction may be more important in practice than correct specification of the diffusion term in pricing bond options.

The paper is organized as follows. Using simulated data, Section 2 shows the bias effects of ML estimation on system parameters, prices of discount bonds and options on a discount bond in the context of a single factor diffusion model. Section 3 introduces a generic version of the jackknife and shows how it can be implemented in parameter estimation, and bond and option valuation. The simulation performance of these jackknife estimates is compared with that of the ML approach. We also discuss bias and variance tradeoffs, consider a version of the jackknife that reduces variability, examine the performance of the jackknife when the model is misspecified, and compare the performance of the jackknife with median unbiased estimation as an alternative method of bias reduction. Section 4 shows the practical effects of jackknifing in an empirical application with monthly and weekly Federal funds rate data. Section 5 concludes and outlines some further applications and implications of the approach.

2 Estimation Bias in Continuous Time Models, Bond Pricing and Bond Option Pricing

We start our discussion with a brief review of some well-known bias results and bias correction methods for discrete time dynamic models. Most relevant in the present context is the fact that standard procedures like ML and least squares (LS) produce downward biased coefficient estimators in the first order autoregression (AR). Using analytic techniques, Hurwicz (1950) demonstrated the bias effect in the first order AR model with known intercept. Using Monte Carlo techniques, Orcutt (1948) and Orcutt and Winokur (1969) found that the bias is larger when the intercept is fitted and explained the bias enlargement in terms of the induced correlation between the regressor and the residual that results from a fitted intercept. Andrews (1993) showed that the presence of a time trend in the regression further accentuates the autoregressive bias. In these two cases, the biases do not go to zero as the AR coefficient goes to zero and the biases increase as the AR coefficient goes to unity.

In the context of the AR(1) model with an intercept only, Kendall (1954) showed that, to a first-order approximation,

$$E[\hat{\phi}] - \phi = -\frac{1+3\phi}{T} + O(\frac{1}{T^2}), \tag{1}$$

where T is the sample size and $\hat{\phi}$ is the ML/LS estimator of the AR coefficient ϕ. A natural bias correction method in this simple setting is

$$\hat{\phi}_K = \hat{\phi} + \frac{1+3\hat{\phi}}{T}. \tag{2}$$

In the finance literature, Bekaert, Hodrick and Marshall (1997) used Kendall's method to correct for bias in testing the expectations hypothesis of the term structure of interest rates. While feasible in this simple model, where (1) and various higher order extensions of (1) have long been known (e.g., Shenton and Johnson, 1965), an undesirable property of the correction method is that it is not directly applicable in more complicated set-ups where asymptotic expansion formulae have not been derived.

As an alternative bias correction method in the AR(1) model with fitted intercept and/or time trend, Andrews (1993) proposed a median unbiased estimator of ϕ. The method relies on knowledge of the exact median function of the estimator. Although the procedure is extended to deal with more general AR(p) models in Andrews and Chen (1994), the estimator is no longer exactly median unbiased and it is not available in more complex models where there are usually additional parameter dependencies in the median function.

Similar bias problems occur in the estimation of continuous time dynamic models. As in discrete time models, the problem is worse when the series are persistent. This phenomenon was documented by Chapman and Pearson (2000), for instance, in the context of the following constant elasticity of variance (CEV) model (c.f. CKLS, 1992),

$$dr(t) = \kappa(\mu - r(t))dt + \sigma r^{\gamma}(t)dB(t), \tag{3}$$

where $B(t)$ is a standard Brownian motion, and $\theta = (\kappa, \mu, \gamma, \sigma)$ is the vector of unknown system parameters. In this model, $r(t)$ mean-reverts towards the unconditional mean μ with speed captured by κ. The observed data are recorded discretely at $(0, \Delta, 2\Delta, \cdots, T\Delta)$ in the time interval $[0, T\Delta]$, where Δ is the step in a sequence of discrete observations of $r(t)$. Since $r(t)$ is often recorded as the annualized interest rate, if it is observed monthly (weekly or daily), we have $\Delta = 1/12$ (1/52 or 1/252).

Chapman and Pearson (2000) used weighted least squares (WLS) to estimate κ in a discretized version of (3) for daily interest rates. Their simulation findings confirm that the estimate of κ is upward biased and that the bias is significant even when the sample size is as large as 7,500. Using the same CEV model, Yu and Phillips (2001) find that alternative Gaussian methods of estimating (3), such as those proposed by Nowman (1997), substantially overestimate κ for daily, weekly and monthly frequencies, whereas the biases are generally small for the other parameters.

These results are not surprising because the CEV model has a discrete time formulation that is very similar to an AR(1) model where there is unconditional heteroscedasticity and with an autoregressive coefficient that is dependent on κ. Since the prices of bonds and bond options also crucially depend on κ, the upward bias in coefficient estimation translates directly into biased bond and option pricing. This important implication of dynamic model estimation bias is explored below in the context of the well-known square-root (CIR) model specialization of (3) due to Cox, Ingersoll and Ross (1985), which is commonly use in practical work and for which there are known closed form options price formulae.

Setting $\gamma = \frac{1}{2}$ in (3), the CIR model has the form

$$dr(t) = \kappa(\mu - r(t))dt + \sigma r^{1/2}(t)dB(t). \tag{4}$$

Feller (1951) and Cox, Ingersoll and Ross (1985) show that the transition density of $r(t+\Delta)$ conditional on $r(t)$ is $ce^{-u-v}(v/u)^{q/2}I_q(2(uv)^{1/2})$ and the marginal density of $r(t)$ is $w_1^{w_2}r^{w_2-1}e^{-w_1 r}/\Gamma(w_2)$, where $c = 2\kappa/(\sigma^2(1 - e^{-\kappa\Delta}))$, $u = cr(t)e^{-\kappa\Delta}$, $v = cr(t)$, $q = 2\kappa\mu/\sigma^2 - 1$, $w_1 = 2\kappa/\sigma^2$, $w_2 = 2\kappa\mu/\sigma^2$, and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. The transition density together with the marginal density can be used for simulation purposes as well as for obtaining the full ML estimator of $\theta(= (\kappa, \mu, \sigma)')$.

Prices of discount bonds and call options on discount bonds based on the square-root model (4) both have analytic solutions. Define $P(t,s)$ as the price at time t of a discount bond that pays-off \$1 at time s and $C(t,\tau;s,K)$ as the value at time t of a call option on a discount bond of maturity data s and of principal L, with exercise (or strike) price K and expiration date τ $(s > \tau > t)$. (Note that, as distinct from options on stock prices, the moneyness of bond options is here determined by the relative size of K to $L\exp(-(s-t)r(t))$; see for example, Buser, Hendershott and Sanders, 1990) Cox, Ingersoll and Ross (1985) show that

$$P(t,s) = A(t,s)e^{-B(t,s)r}, \tag{5}$$

and that

$$\begin{aligned} C(t,\tau;s,K) \quad = \quad & LP(t,s)\chi^2(2r^*(\phi+\psi+B(\tau,s)); \frac{4\kappa\mu}{\sigma^2}, \frac{2\phi^2 re^{\gamma(\tau-t)}}{\phi+\psi+B(\tau,s)}) \\ & -KP(t,\tau)\chi^2(2r^*(\phi+\psi); \frac{4\kappa\mu}{\sigma^2}, \frac{2\phi^2 re^{\gamma(\tau-t)}}{\phi+\psi}), \end{aligned} \tag{6}$$

where

$$
\begin{aligned}
A(t,\tau) &= (\frac{2\gamma e^{(\kappa+\gamma)(\tau-t)/2}}{(\kappa+\gamma)(e^{\gamma(\tau-t)}-1)+2\gamma})^{2\kappa\mu/\sigma^2},\\
B(t,\tau) &= \frac{2(e^{\gamma(\tau-t)}-1)}{(\kappa+\gamma)(e^{\gamma(\tau-t)}-1)+2\gamma},\\
\gamma &= \sqrt{\kappa^2+2\sigma^2},\\
\phi &= \frac{2\gamma}{\sigma^2(e^{\gamma(\tau-t)}-1)},\\
\psi &= (\kappa+\gamma)/\sigma^2,\\
r^* &= \ln(A(\tau,s)/k)/B(\tau,s),\\
k &= K/L,
\end{aligned}
$$

and $\chi^2(\cdot;q,p)$ is the cumulative distribution function of a noncentral chi-square variate with q degrees of freedom and the noncentrality parameter p.

It is clear from equations (5) and (6) that both bond and option prices depend on the mean reversion parameter, κ. Fig. 2 plots the price of a discount bond and the price of the option on the discount bond as a function of κ. The discount bond is a three-year bond (hence $t = 0$ and $s = 3$) with a face value of \$1 and initial interest rate of 5%. The one-year European call option on a three-year discount bond has a face value of \$100 and a strike price of \$87 (ie $t = 0, r(t) = 0.05, s = 3, \tau = 1, L = 100, K = 87$). These parameters are empirically reasonable and imply that $K/(L\exp(-(s-t)r(t))) = 1.011$. Hence, in this case we have to price an out-of-the-money option.

We choose $\mu = 0.08, \sigma = 0.02$ in model (4). It can be seen that as κ changes both bond and option prices change in a nonlinear and monotonically decreasing fashion. So any bias in estimated κ is transmitted to the corresponding estimates of the bond and option prices. In particular, overestimation of κ leads to underestimation of the bond and option prices. Also, as is apparent in Fig. 2, the bond price is much less sensitive to a change in κ than the option price. As a result, we expect bias in κ to have a larger impact on option pricing. Furthermore, in both cases the sensitivity depends on the magnitude of κ. The smaller is κ, the larger the sensitivity.

Figs. 3-6 plot, in the percentage terms, the bias of the ML estimator of κ, and the bias of the estimated bond and option prices (using plugged in

ML estimates) as a function of κ. The results given in these figures are based on simulations which allow for both monthly and weekly frequencies and the following range of possible true values of $\kappa : 0.1, 0.15, 0.2, 0.25, 0.3, 0.4$. Figs. 3-4 give the results for the monthly frequency with sample sizes equal to 300 and 600. Figs. 5-6 give results for the corresponding weekly frequency with sample sizes equal to 1000 and 2000. All results are based on 1000 replications.

The following general conclusions emerge from these results. First, the ML estimator of κ is upward biased and the percentage bias decreases monotonically with the true value of κ. This result is consistent with what is known about dynamic bias in AR/unit root models (e.g., Andrews, 1993), as larger κ corresponds to a smaller AR coefficient. In all cases, the biases are serious for empirically relevant values of κ and sample sizes.

Second, although the bias in the ML estimator of the parameter κ is serious, this bias does not translate into a serious bias for the bond price. The outcome is partly explained by Fig. 2, where it is clear that the bond price is not very sensitive to changes in κ. However, bonds are always under priced and this is consistent with the upward bias in estimated κ. In magnitude, the bias monotonically decreases with the true value of κ and stays within the 2% range.

By contrast, the options price is substantially underestimated. The percentage bias is generally non-monotonic in κ and in all cases considered it is larger than 20%. In some cases, the bias in the options price is as high as 45% for monthly data and 52% for weekly data. Hence, bond options are significantly underpriced when κ is estimated by ML. The most serious bias occurs when κ takes values in the interval $[0, 0.3]$, which is empirically the most relevant range of κ.

Finally, the bias of the ML estimator of κ, and the bias in bond and option prices all get smaller as the sample size increases. This means that on average ML traders would increase the option price when more observations are available. Nonetheless, the bias in κ and the bias in option prices are still nonnegligible even for large sample sizes. These results indicate that biases in the estimation of these quantities must be expected to occur in practical work where the empirical sample sizes are in the same general range as those considered here. The biases are particularly problematic in the case of bond options prices.

3 Jackknife Estimation of System Parameters, Bond Prices and Option Prices

3.1 Jackknife estimation

Quenouille (1956) proposed the jackknife as a solution to finite sample bias in parametric estimation problems. Let T be the number of observations in the whole sample and decompose the sample into m consecutive sub-samples each with ℓ observations, so that $T = m \times \ell$. The jackknife estimator of a certain parameter, θ, then utilizes the subsample estimates of θ to assist in the bias reduction process giving

$$\hat{\theta}_{jack} = \frac{m}{m-1}\hat{\theta}_T - \frac{\sum_{i=1}^{m} \hat{\theta}_{\ell i}}{m^2 - m}, \tag{7}$$

where $\hat{\theta}_T$ and $\hat{\theta}_{\ell i}$ are the estimates of θ obtained by application of a given method like ML to the whole sample and the i'th sub-sample, respectively. Under quite general conditions which ensure that the bias of the estimates $(\hat{\theta}_T, \hat{\theta}_{\ell i})$ can be expanded asymptotically in a series of increasing powers of T^{-1}, it can be shown that the bias in the jackknife estimate $\hat{\theta}_{jack}$ is of order $O(T^{-2})$ rather than $O(T^{-1})$.

The result can be demonstrated as follows using Sargan's (1976) theorem on the validity of the (Nagar) approximation of the moments of statistical estimator in terms of the moments of the estimator's Taylor expansion as a polynomial of more basic statistics (like sample moments of the data). To fix ideas, suppose $\hat{\theta}_T = \theta_T(p_T)$, where p_T is an $N-$ vector of sample moments of the data with mean μ, whose Taylor development to order k is valid and has the form

$$\theta_{T,k}(p_T) = \sum_{s=0}^{k-1} \frac{1}{s!} \left[\left\{ (p_T - \mu)' \frac{\partial}{\partial p} \right\}^s \theta_T(p) \right]_{p=\mu}. \tag{8}$$

It is frequently the case in practical applications that $p_T - \mu = O_p\left(T^{-\frac{1}{2}}\right)$ and then (8) produces a corresponding stochastic expansion. Under some mild regularity conditions on the derivatives of $\theta_T(p_T)$ that appear in (8) and the order of magnitude of the moments of $\hat{\theta}_T$ and p, which are assumed to exist, Sargan (1976, Theorems A1 & A2) proved that

$$E\left(|\theta_T(p)|^j\right) = E\left(|\theta_{T,k}(p)|^j\right) + O\left(T^{-\gamma k}\right), \quad \gamma > 0, \tag{9}$$

so that for suitably large k, we can replace the j'th moment of $\hat{\theta}_T$ by the j'th moment of the polynomial approximation $\theta_{T,k}(p)$. This theorem holds rather generally and applies in the present context where $\hat{\theta}_T$ is an econometric estimator of the parameters in the diffusion equation (4) and p_T is a vector of sample moments of discrete data generated by the model (4). The functional dependence $\hat{\theta}_T = \theta_T(p_T)$ and its Taylor representation (8) may also be obtained indirectly. In the case of extremum estimators like ML, this involves the use of the implicit function theorem and power series inversion of the Taylor expansion of the first order conditions.

When $\hat{\theta}_T$ is a consistent estimator of θ and when the moment expansion $E(p_T) = \mu + \frac{b_1}{T} + O(T^{-2})$ holds for some constant b_1, we can apply (9) to deduce that for some constant a_1

$$E(\hat{\theta}_T) = \theta + \frac{a_1}{T} + O(\frac{1}{T^2}), \text{ and } E(\hat{\theta}_{\ell i}) = \theta + \frac{a_1}{T/m} + O(\frac{1}{(T/m)^2}. \tag{10}$$

Taking expectations in (7) and substituting the two expressions in (10) leads directly to the expansion

$$\begin{aligned} E(\hat{\theta}_{jack}) &= \frac{m}{m-1}\theta + \frac{m}{m-1}\frac{a_1}{T} - \frac{\sum_{i=1}^{m}(\theta + \frac{a_1}{T/m})}{m^2 - m} + O(\frac{1}{T^2}) \\ &= \theta + O(\frac{1}{T^2}), \end{aligned} \tag{11}$$

reducing the $O(T^{-1})$ bias (10) in the unmodified estimate $\hat{\theta}_T$ to $O(T^{-2})$ in $\hat{\theta}_{jack}$. Note that (11) is invariant to the choice of m to $O(T^{-1})$.

In view of the generality of (9), this bias reduction procedure can be expected to be widely applicable. It is also very easy to implement in practical work. In the present case, we assume that the above theory applies, validating (10) and (11). The quantity θ can be either a parameter (such as κ), a function of parameters (such as the bond option price) or a vector of several such quantities. In the case of bond and options prices, θ will depend on known variables such as t, s and τ, as well as unknown parameters such as κ. These additional dependencies do not affect the validity of the procedure.

In the context of square-root diffusions, we propose to jackknife not only the parameter κ, but also the bond and option prices directly. We have found that there is substantial advantage to the latter procedure of dealing directly with the quantity of interest in implementing the jackknife rather jackknifing the parameter estimates on which the option price depends and plugging this revised estimate into the options price formula. The reason is that the jackknife tends to increase the variance of the quantity being estimated and this additional variance adversely affects the performance of the procedure when the quantity is a very nonlinear function of its arguments, like the option price. In such cases, it appears to be much better to apply the jackknife directly to estimate the option price (see Section 3.4).

In implementing the jackknife (7), it is often convenient to choose $m = 2$ (two subsamples) and this simple choice has very satisfactory performance in bias reduction. In the simulations reported below, we also tried the value $m = 4$ and there are certain advantages to increasing the value of m. In particular, while the mean expansion (11) is invariant to m to order T^{-1}, the variability of $\hat{\theta}_{jack}$ depends on m, as is apparent from the following expression for the scaled estimation error of $\hat{\theta}_{jack}$:

$$\sqrt{T}\left(\hat{\theta}_{jack}-\theta\right)=\left(1+\frac{1}{m-1}\right)\sqrt{T}\left(\hat{\theta}_{T}-\theta\right)-\frac{1}{m-1}\left\{\frac{1}{\sqrt{m}}\sum_{i=1}^{m}\sqrt{\ell}\left(\hat{\theta}_{\ell i}-\theta\right)\right\}. \tag{12}$$

In (12) $\sqrt{T}$ scaling is presumed to be appropriate for $\hat{\theta}_T$ and for $\hat{\theta}_{jack}$, and analogous formulae would apply in the case where there happened to be a faster convergence rate (e.g. due to nonstationarity). It might be anticipated from this expression [1] that larger values of m may help to reduce the variation of $\hat{\theta}_{jack}$ and, therefore, since (11) still holds, the RMSE of the jackknife estimator $\hat{\theta}_{jack}$. These heuristics are supported in the present case by the simulation results, which reveal that use of $m = 4$ enables both bias reduction and MSE reduction in estimation.

[1]For example, if $\frac{1}{l}+\frac{m}{l}\to 0$ as $T\to\infty$, if the data are weakly dependent, and if the estimates $\hat{\theta}_{li}$ and $\hat{\theta}_T$ are asymptotically normally distributed as

$$\sqrt{l}\left(\hat{\theta}_{li}-\theta_{li}\right),\sqrt{T}\left(\hat{\theta}_{T}-\theta\right)\to_d N\left(0,\sigma_\theta^2\right),$$

then, in general,

$$\sqrt{T}\left(\hat{\theta}_{jack}-\theta\right)\to_d N\left(0,\sigma_\theta^2\right),$$

also. However, as m increases, higher order terms in the expansions suggest that the finite sample variation of $\sqrt{T}\left(\hat{\theta}_{jack}-\theta\right)$ decreases with m. A detailed examination of these issues will be provided in later work.

Since full ML estimation of (4) is feasible for the square-root diffusion, the jackknife procedure can be based on ML. The following specific steps were involved in the implementation of the procedure.[2]

1. Estimate the system parameters by ML using the entire sample.
2. Calculate the bond and option prices based on the ML estimates obtained in Step 1.
3. Estimate the system parameters by ML for each sub-sample.
4. Calculate the bond and option prices based on the ML estimates obtained in Step 3 for each sub-sample.
5. Calculate the jackknife estimators of κ, and the bond price and option prices using equation (7).

To compare the performance of the jackknife and ML estimators, we use the same Monte Carlo experiments as in the previous section. We first set m to 2. Data are simulated from a square-root model with $\mu = 0.08$, $\sigma = 0.02$ and $\kappa = 0.1, 0.15, 0.2, 0.25, 0.3, 0.4$. For monthly data we choose $T = 300$, 600 and for weekly data we choose $T = 1000$, 2000. The number of replications is 1000. The discount bond is a three-year bond with a face value of $1 and initial interest rate of 5%. The one-year European call option on a three-year discount bond has a face value of $100 and a strike price of $87.

Figs. 7-10 plot, in percentage terms, the biases of the jackknife estimator of κ, and the bond and option prices as functions of κ. We also plot biases of the ML estimator for comparison. Figs. 7-8 correspond to the monthly frequency with sample sizes $T = 300$, 600. Figs. 9-10 correspond to the weekly frequency with sample sizes $T = 1000$, 2000. In all figures, the solid line represents the ML estimate while the marked line represents the jackknife estimate.

First, it can be seen that the jackknife procedure successfully reduces the bias in the estimation of κ across all cases. The jackknife works surprisingly

[2] *Matlab* code to implement the procedure in the context of square-root diffusions can be found at

http://yoda.eco.auckland.ac.nz/~jyu/research.html.

This code covers both the simulations and the empirical work discussed below.

well even when the sample size is small. Also, the improvement over ML is greater when the true value of κ is smaller, which is the more relevant case in empirical work. Second, although we have already found that ML estimated bond price has only a small downward bias, the jackknife estimated bond price still produces gains in all cases. The marginal difference between these estimates is a decreasing function of κ. Third, and most importantly, we find the jackknife estimated bond option price is substantially better than ML. The bias reduction from the jackknife is at least 12%, 8%, 10% and 15% across the four cases. These gains are of sufficient magnitude to make an important difference in practical work. Fourth, the jackknife method still appears to underprice the option.

Tables 1-4 compare the means, standard deviations, and RMSE's of the MLE of κ and the ML estimated option price and bond price with those obtained by the jackknife in the same experiments. Tables 1-2 give the monthly frequency results with sample sizes 300 and 600, while Tables 3-4 give the weekly frequency results with sample sizes 1000 and 2000. Consistent with Figs. 7-10, the jackknife method is seen to provide a significant improvement in terms of bias reduction over ML in the estimation of κ and option prices, while the gains in estimating bond prices are marginal but still uniform across all cases. The bias reductions from the jackknife in pricing options are achieved at the cost of a minor increase in RMSE.

3.2 Variance reduction

All of these results refer to the case where $m = 2$ and the jackknife is based on only two subsamples. In this case it is apparent from the findings that there is a trade-off between the (often substantial) bias reduction achieved by the jackknife and a marginal increase in the dispersion of the estimates. As argued above, it is possible to reduce the variability of the jackknife estimate (with a small compromise in the bias reduction gains) by using larger values of m. To illustrate the effectiveness of this approach, we use the same Monte Carlo design as before and focus on the empirically more relevant case where $\kappa = 0.1$, considering in all the following four cases: $T = 300$, 600 monthly observations, and $T = 1000$, 2000 weekly observations. Table 5 compares the means, standard deviations, and RMSE's of the MLE of κ and bond and option prices with those obtained from jackknife estimates with $m = 4$ in (7).

It can be seen that the jackknife provides smaller RMSE than ML in all cases and continues to achieve major bias reductions. However, m cannot be set too large because the subsample estimates of θ rely on $\ell = T/m$ observations and ℓ needs to be large enough to ensure that the subsample likelihood has a well behaved optimum.

It is noteworthy that, on average, the jackknife method underestimates κ as well as the option price. At first glance the direction of the bias in the option price estimates seems inconsistent with the direction of the bias in κ. However, since the option price is a nonlinear transformation of κ, an underestimated κ has a different impact in magnitude on the option price from an overestimated κ. As a result, although the jackknife estimate of κ has little asymmetry (as is apparent in Fig. 11), the jackknifed option price becomes asymmetric (as is apparent in Fig. 1). Obviously the nonlinearity is the cause of this counter-intuitive result.

3.3 Specification bias versus estimation bias

Specification issues in continuous time modelling of short time interest rates have been a focal point of much recent literature in finance. Important contributions include CKLS (1992) and Aït-Sahalia (1996a, 1996b), Stanton (1997), Bandi and Phillips (2002), Bandi (2002), and Hong and Li (2002), to mention only a few. Using the CEV model, for example, CKLS (1992) reject all more restricted nested single factor models and find that the CEV model leads to option prices that are significantly different from those implied by simpler interest rate processes.

In view of the importance of the diffusion specification to option pricing, it is of interest to compare the magnitude of estimation bias in the drift to bias effects arising from diffusion misspecification. The relative importance of these two effects can be assessed in simulation. To do so, we simulate 600 monthly observations from the following model (Vasicek, 1977),

$$dr(t) = \kappa(\mu - r(t))dt + \sigma dB(t), \tag{13}$$

where $(\kappa, \mu, \sigma)'$ is set at $(0.1, 0.12, 0.015)'$.[3] The discount bond is a three-year

[3] We use slightly different parameter values here (specifically, $\mu = 0.12$ and $\sigma = 0.015$ rather than $\mu = 0.08$ and $\sigma = 0.02$) because data from the Vasicek model can become negative and these parameter settings avoid negative values in the 1000 replications used in this comparison.

bond with a face value of \$1 and initial interest rate of 5%. The one-year European call option on a three-year discount bond has a face value of \$100 and various strike prices so that the ratio of the strike price to the current value of par takes each of the following values: $0.95, 1, 1.05$. These ratios correspond to in-the-money, at-the-money, and out-of-the-money situations, respectively. The number of replications is 1000. Each simulated sequence is fitted under the (misspecified) CIR model to obtain the ML and jackknife estimates of κ, the bond price and option price.

Since the diffusion term is misspecified in estimating the CIR model, the ML estimates are biased. However the exact discrete model corresponding to (13) is (Phillips, 1972)

$$r(t) = e^{-\kappa\Delta} r(t-\Delta) + \mu\left(1 - e^{-\kappa\Delta}\right) + \sigma \int_{t-\Delta}^{t} e^{-\kappa(t-s)} dB(s), \tag{14}$$

whose autoregressive term is the same as that of the discrete model corresponding to a CIR model. Therefore, we may expect that ML estimates of the drift function in the misspecified model continue to suffer from dynamic estimation bias, making the jackknife desirable. Of course, the ML estimates of the drift function in the correctly specified discrete model (14) will also suffer from dynamic estimation bias. The experimental design in the simulation enables us to isolate the bias arising in the estimation of the drift from that due to misspecification of the diffusion.

Tables 6-8 compare the means, standard deviations, and RMSE's of the ML and jackknife estimates of κ, the bond price, and the option price, true values of these quantities also being shown for comparison purposes. The true bond value is obtained using the analytic formula given in Vasicek (1977) and the true option value is calculated based on the analytic formula derived by Jamshidian (1989). Tables 6-8 report results for ML estimation of the correctly specified (Vasicek) model obtained from the exact discrete model (14), ML estimation of the CIR model where the diffusion function is misspecified, and jackknife estimates based on the misspecified CIR model.

It is clear from Tables 6-8 that the bias effect plays an important role in all cases. For example, in comparing the ML and jackkife esimates of the misspecified CIR model when $m = 2$, the jackknife method reduces the bias in the bond price from -1.73% to -0.27%, and the bias in option prices from -21.04% to -5.89%, -40.02% to -23.94%, and -57.65% to -47.72% for the in-the-money, at-the-money, and out-of-the-money options respectively. These are substantial improvements, indicating that the jackknife continues to be a very effective tool of bias reduction even in misspecified situations. When $m = 4$, the jackknife method reduces the bias in the bond price from -1.73% to -0.51%, and the bias in option prices from -21.04% to -10.27%, -40.02% to -30.05%, and -57.65% to -50.34% for the in-the-money, at-the-money, and out-of-the-money options respectively, while also achieving reductions in RMSE over the $m = 2$ setting. Finally, in comparing the ML estimates of the correctly specified (Vasicek) model with the jackknife estimates ($m = 4$) of the misspecified CIR model, we find that the jackknife continues to reduce the bias in the bond price, now from -1.83% to -0.51% and the bias in options prices from -21.14% to -10.27% and from -36.15% to –29.85% for the in-the-money and at-the-money cases. Only in the out-of-the-money case does ML have lower bias in the correctly specified model for the option price than the jackknife estimate from the misspecified model. The improvements for the bond price and for the in-the-money and at-the-money options indicate that the bias arising from estimating the drift term is generally more serious than that arising from misspecification of the diffusion.

3.4 Jackknife versus median unbiased estimation

In addition to the jackknife, median unbiased estimation (MUE) introduced by Andrews (1993) can also remove bias in coefficient estimation of the discrete AR(1) model. This method relies on full knowledge of the exact median function of the estimator, which can only be obtained by simulation. For continuous time models with nonlinear diffusions the situation is exacerbated because the distribution is non Gaussian, there is conditional heterogeneity in the model, and the median function must be approximated using a computationally intensive Monte Carlo method that depends on specific values of the other parameters in the model. In consequence, MUE is not a practically feasible method.

In spite of these practical limitations, the MUE procedure provides a very interesting benchmark for evaluating the success of bias reduction procedures in dynamic models. The MUE is obtained by transforming the ML estimate $\hat{\kappa}$ with the inverse median function m_d^{-1}, where the median function $m_d(\kappa)$ is found by running extensive simulations over a wide range of parameter values κ. In this way the MUE procedure utilizes a great deal of information about the distribution of the ML estimator and, at least when the assumptions underlying the construction of m_d are valid, we can expect this bias reduction procedure to be hard to beat.

In the present study, we implemented the MUE in the CIR model, where its performance can be directly compared with that of the jackknife and ML. The median functions (for various sample sizes) of the ML estimate of κ in the CIR model were obtained by simulation.[4] Using these simulated median functions, we correct the bias by constructing the median unbiased estimator of κ as in Andrews (1993). Fig. 12 compares the density of the estimates of κ from ML, the jackknife with $m = 4$ and MUE, where 300 monthly observations are used to estimate the CIR model for parameter values $\kappa = 0.1$, $\mu = 0.08$, $\sigma = 0.02$, and the number of replications is 1000. It can be seen that the jackknife method works nearly as well as MUE in terms of correcting for estimation bias in κ. Table 9 reports the means, standard deviations, and RMSE's of estimate of κ, bond price and option price from ML, jackknife with $m = 2, 4$, and MUE, where the bond and bond option are defined in the same way as before. For comparison purposes, we also plug-in the jackknife parameter estimates to the bond and option price formulae and report the results in Table 9. In terms of the bias reduction in κ, MUE is comparable to jackknife but has smaller RMSE. The results suggest that MUE is very effective in reducing dynamic bias, as indeed it is designed to do. Not surprisingly, the plug-in MUE works better than the plug-in jackknife for pricing bonds and options as it provides smaller biases as well as smaller RMSE's. However, the plug-in MUE is not superior to the estimates obtained from jackknifing the quantities of interest directly. Although, in magnitude, the bias in the bond and option prices from MUE is comparable to that from jackknifing bond and option prices with $m = 2$, the RMSE is larger for MUE, particularly for option prices.

These results confirm that the idea of jackknifing specific quantities of interest directly, rather than plugging in bias reduced parameter estimates is likely to be especially important in practice, as in the present setting where the object is to achieve gains in pricing derivative securities where the formulae are complicated functions of several fundamentals including unknown parameters. It is a characteristic feature of the jackknife method that it permits corrections to be implemented directly on the ultimate quantity of

[4]The median functions may be obtained from the authors upon request.

interest. This is an important operational distinction between the jackknife and other bias reduction techniques like MUE which have more limited applicability in view of distributional and invariance property restrictions.

The reason jackknifing option prices directly works better than various plug-in methods (including the plug-in jackknife) is because of the nonlinear nature of pricing applications. It is already well known that nonlinearity can cause plug-in methods to produce biased option price estimates even when the parameter estimate is unbiased (Butler and Schachater, 1986). Our results demonstrate that bias effects can be much more significant when the parameter estimate itself is biased and the pricing function is heavily nonlinear. The nonlinearity of the options pricing formula has other important implications in finance. For instance, when comparing alternative option valuation models, Christoffersen and Jacobs (2002) found that a method can often perform well out-of-sample in a dimension that corresponds to the loss function applied in estimating or calibrating the forecasts, but not necessarily well in other dimensions. These differences in option pricing performance are largely driven by the nonlinearities of option pricing functions. By avoiding the use of plug-in operations entirely, appropriate use of the jackknife can be expected to be more robust against the effects of nonlinearity.

4 Empirical Application

This empirical application compares the estimated κ, bond prices, and option prices implied by the MLE with those of the jackknifed MLE in the case of two short term interest rate series. Both datasets involve the Federal funds rate and are available from the H-15 Federal Reserve Statistical Release. The first is sampled monthly and has 576 observations covering the period from July 1954 to June 2002. The second is sampled weekly and has 2112 observations from July 7, 1954 to December 21, 1994. The sample sizes are chosen to be close to those used in the simulation study to help in calibrating the results with the simulation. Since all yields are expressed in annualized form, we have $\Delta = 1/12$ for the monthly data and $\Delta = 1/52$ for the weekly data.

Time series plots of the two datasets are provided in Fig. 13. Table 10 shows the sample sizes, means, standard deviations, first seven autocorrelations, and Phillips' (1987) $Z(t)$ unit root test statistic (with a fitted intercept in the regression) for both series. The presence of a unit root is rejected at the 10% level but cannot be rejected at the 5% level in both series. These results, together with the form of the sample autocorrelogram, suggest that both interest rate series are highly persistent. Hence, standard estimation methods for diffusion equations can be expected to lead to significant bias in estimating the correlation and mean reversion coefficients.

In this comparison, we focus on a three-year discount bond, a one-year call option, and a half-year call option on a discount bond. In all cases the initial value for the short term interest rate is set at 6%. For the discount bond we choose a face value of \$1. The call option on a three-year discount bond has a face value of \$100 and various strike prices so that the ratio of the strike price to the current value of par takes each of the following values: $0.95, 1, 1.05$. These ratios correspond to in-the-money, at-the-money, and out-of-the-money situations, respectively. We first estimate the square-root model by ML and jackknife methods and then obtain the ML estimated bond and option prices as well as the jackknife estimates of the two prices.

Tables 11 and 12 report the results with $m = 2$. For monthly data, the jackknife estimate of κ is 0.0846 and is 16.9% smaller than the MLE; the jackknife estimate of the bond price is 0.3% higher than its ML counterpart. More importantly, the call option values differ substantially between the two methods. The biggest percentage differences are for the out-of-the-money option. For example, for the half-year option, the price of the out-of-the-money option implied by the jackknife method is 13% larger than that obtained from the MLE. For weekly data, the jackknife estimate of κ is 0.2222 and is 37.4% smaller than the MLE; and the jackknife estimate of the bond price is 0.02% higher than its ML counterpart. For call option values, the differences are even bigger than in the case of monthly data. For example, for the half-year option, the price of the out-of-the-money option implied by the jackknife method is 21% larger than that by the ML method. All these results are consistent with the magnitudes and directions of the biases and differences between the jackknife and ML estimates that were found in the simulation studies.

5 Conclusions and Implications

Bias in the estimation of the parameters of continuous time models by standard methods such as ML translates into bias in pricing bonds and bond options. In cases where the parameters take on realistic values, we have found that these biases can be substantial, particularly in the case of bond options. The procedure we propose here for reducing the bias involves the use of subsample estimates and a version of the jackknife. Simulations show the procedure to be highly effective in a CIR model and to offer substantial improvements in pricing bond options and marginal improvements in pricing bonds over the usual ML approach. The greatest gains are in the substantial bias reductions that the jackknife method provides. But use of multiple subsamples in the construction of the jackknife enables reductions in both bias and mean squared error, so the gains from bias reductions are not lost in variance increases. An interesting feature of the proposed method is that it can be used to reduce bias even when the diffusion of the model is misspecified, thereby offering an additional advantage over standard methods. Our simulation findings indicate that the dynamic estimation bias arising from the use of standard estimation methods can be even larger than the specification bias arising from misspecifying the diffusion. Moreover, using the jackknife in a model where the diffusion is misspecified turns out to be less biased than using ML in the correctly specified model.

The present paper applies the approach to price discount bonds and options in the context of a square-root diffusion estimated by MLE. Use of this specific model makes it possible to employ full ML and, importantly, closed form options price formulae. So precise evaluation of the performance of the jackknife procedure is possible in this set up. Nonetheless, the technique itself is quite general and can be applied in many other contexts and models with little modification. For example, the method extends to a broader range of model specifications, including the CEV model (CKLS, 1992), extended one-factor models (Hull and White, 1990), two-factor equilibrium models (Brennan and Schwartz, 1979, 1982, and Longstaff and Schwartz, 1992, Langetieg, 1980, and Countadon, 1982), the semiparametric model (Aït-Sahalia 1996) with parametric drift, models with stochastic volatility (Andersen and Lund, 1997) and diffusion models with jumps (Duffie, Pan and Singleton, 2000). A particularly interesting class of term structure models is the multifactor affine family that has been studied intensively in the recent literature for pricing interest rate derivatives (e.g., Duffie and Kan, 1996, and Dai and Singleton, 2000). Because the drift is linear and parametric in this family, we expect the bias problem to be present in these models also and again our method should be useful.

In more complicated models such as those just mentioned, the analytic form of the likelihood function is often unavailable and so exact ML is infeasible. However, the proposed jackknife method can be used in connection with other estimation methods. Examples include simulated GMM (Duffie and Singleton, 1993), EMM (Gallant and Tauchen, 1996), indirect inference (Gourieroux, Monfort and Renault, 1996), continuous time GMM (Hansen and Scheikman, 1995), approximate ML (Aït-Sahalia, 2002), simulated ML (Pedersen, 1995, Brandt and Santa-Clara, 2002, Durham and Gallant, 2002), and methods via the empirical characteristic function (Singleton, 2001, and Knight and Yu, 2002). Finally, many other interest-rate-contingent claims can be treated in a similar way. Examples include coupon-bearing bonds, caps, swaptions, captions, mortgage-back securities, and stock and currency options with stochastic interest rates. Since all these interest-rate-contingent claims are nonlinear functions of the system parameters, any bias in the estimation of the system parameters will carry over to pricing the interest-rate-contingent claims. The situation in these cases is analogous to the one explored here and the proposed jackknife method can be used to reduce the bias in pricing the contingent claims.

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Table 1. Monte Carlo study comparing mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond prices and option prices when the number of monthly observations is 300

	ML Estimates			Jackknife Estimates		
	κ	Bond	Option	κ	Bond	Option
True Value	**0.1**	**0.8503**	**2.3921**	**0.1**	**0.8503**	**2.3921**
Mean	0.2762	0.8381	1.4071	0.0988	0.8460	1.8186
Std Dev	0.1853	0.0125	0.8959	0.2606	0.0163	1.2555
RMSE	0.2557	0.0174	1.3315	0.2606	0.0169	1.3803
True Value	**0.15**	**0.8458**	**1.9955**	**0.15**	**0.8458**	**1.9955**
Mean	0.3267	0.8349	1.1468	0.1342	0.8426	1.5043
Std Dev	0.2007	0.0118	0.8182	0.2736	0.0157	1.1681
RMSE	0.2674	0.0161	1.1789	0.2740	0.0160	1.2672
True Value	**0.2**	**0.8418**	**1.6315**	**0.2**	**0.8418**	**1.6315**
Mean	0.3723	0.8321	0.9041	0.1891	0.8389	1.1808
Std Dev	0.2035	0.0110	0.7466	0.2800	0.0147	1.0724
RMSE	0.2666	0.0147	1.0424	0.2802	0.0150	1.1633
True Value	**0.25**	**0.8381**	**1.2918**	**0.25**	**0.8381**	**1.2918**
Mean	0.4156	0.8360	0.7170	0.2313	0.8360	0.9425
Std Dev	0.2151	0.0140	0.6777	0.2915	0.0140	0.9943
RMSE	0.2714	0.0141	0.8886	0.2921	0.0141	1.0539
True Value	**0.3**	**0.8348**	**0.9735**	**0.3**	**0.8348**	**0.9735**
Mean	0.4653	0.8273	0.5550	0.2795	0.8330	0.7126
Std Dev	0.2265	0.0100	0.5954	0.2909	0.0133	0.8729
RMSE	0.2804	0.0125	0.7277	0.2916	0.0134	0.9111
True Value	**0.4**	**0.8290**	**0.4370**	**0.4**	**0.8290**	**0.4370**
Mean	0.5728	0.8227	0.2953	0.3816	0.8273	0.3507
Std Dev	0.2441	0.0092	0.4314	0.3082	0.0120	0.6340
RMSE	0.2991	0.0111	0.4540	0.3087	0.0121	0.6399

Table 2. Monte Carlo study comparing mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond prices and option prices when the number of monthly observations is 600

	ML Estimates			Jackknife Estimates		
	κ	Bond	Option	κ	Bond	Option
True Value	**0.1**	**0.8503**	**2.3921**	**0.1**	**0.8503**	**2.3921**
Mean	0.1845	0.8437	1.8085	0.0933	0.8492	2.2068
Std Dev	0.1013	0.0080	0.6920	0.1397	0.0104	0.9144
RMSE	0.1319	0.0103	0.9052	0.1399	0.0105	0.9330
True Value	**0.15**	**0.8458**	**1.9955**	**0.15**	**0.8458**	**1.9955**
Mean	0.2336	0.8400	1.4765	0.1462	0.8447	1.7858
Std Dev	0.1092	0.0078	0.6700	0.1451	0.0100	0.8744
RMSE	0.1375	0.0098	0.8475	0.1452	0.0101	0.8992
True Value	**0.2**	**0.8418**	**1.6315**	**0.2**	**0.8418**	**1.6315**
Mean	0.2842	0.8364	1.1649	0.1994	0.8405	1.4069
Std Dev	0.1210	0.0078	0.6467	0.1564	0.0098	0.8547
RMSE	0.1474	0.0095	0.7974	0.1564	0.0099	0.8837
True Value	**0.25**	**0.8381**	**1.2918**	**0.25**	**0.8381**	**1.2918**
Mean	0.3314	0.8333	0.9026	0.2479	0.8360	1.0861
Std Dev	0.1275	0.0076	0.5977	0.1586	0.0140	0.7950
RMSE	0.1513	0.0090	0.7133	0.1587	0.0141	0.8212
True Value	**0.3**	**0.8348**	**0.9735**	**0.3**	**0.8348**	**0.9735**
Mean	0.3810	0.8306	0.6825	0.2933	0.8340	0.8172
Std Dev	0.1358	0.0072	0.5268	0.1690	0.0089	0.7039
RMSE	0.1582	0.0084	0.6018	0.1697	0.0090	0.7210
True Value	**0.4**	**0.8290**	**0.4370**	**0.4**	**0.8290**	**0.4370**
Mean	0.4819	0.8256	0.3399	0.3957	0.8282	0.3755
Std Dev	0.1485	0.0066	0.3743	0.1774	0.0079	0.4981
RMSE	0.1696	0.0075	0.3867	0.1775	0.0080	0.5019

Table 3. Monte Carlo study comparing mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond prices and option prices when the number of weekly observations is 1000

	ML Estimates			Jackknife Estimates		
	κ	Bond	Option	κ	Bond	Option
True Value	**0.1**	**0.8503**	**2.3921**	**0.1**	**0.8503**	**2.3921**
Mean	0.3270	0.8357	1.2827	0.0733	0.8452	1.7201
Std Dev	0.2371	0.0144	0.9322	0.3463	0.0187	1.3456
RMSE	0.3282	0.0205	1.4491	0.3473	0.0194	1.5041
True Value	**0.15**	**0.8458**	**1.9955**	**0.15**	**0.8458**	**1.9955**
Mean	0.3827	0.8325	1.0330	0.1403	0.8408	1.3913
Std Dev	0.2494	0.0137	0.8544	0.3429	0.0181	1.2346
RMSE	0.3411	0.0191	1.2870	0.3431	0.0188	1.3745
True Value	**0.2**	**0.8418**	**1.6315**	**0.2**	**0.8418**	**1.6315**
Mean	0.4368	0.8295	0.7995	0.1972	0.8371	1.0707
Std Dev	0.2608	0.0128	0.7801	0.3623	0.0174	1.1444
RMSE	0.3522	0.0178	1.1405	0.3623	0.0180	1.2744
True Value	**0.25**	**0.8381**	**1.2918**	**0.25**	**0.8381**	**1.2918**
Mean	0.4784	0.8270	0.6261	0.2377	0.8339	0.8288
Std Dev	0.2599	0.0123	0.6913	0.3571	0.0167	1.0333
RMSE	0.3460	0.0166	0.9597	0.3573	0.0172	1.1323
True Value	**0.3**	**0.8348**	**0.9735**	**0.3**	**0.8348**	**0.9735**
Mean	0.5301	0.8249	0.4924	0.2959	0.8310	0.6456
Std Dev	0.2822	0.0116	0.5979	0.3818	0.0158	0.8997
RMSE	0.3641	0.0152	0.7674	0.3819	0.0163	0.9576
True Value	**0.4**	**0.8290**	**0.4370**	**0.4**	**0.8290**	**0.4370**
Mean	0.6175	0.8213	0.2742	0.3809	0.8264	0.3381
Std Dev	0.2945	0.0101	0.4265	0.3835	0.0134	0.6380
RMSE	0.3661	0.0127	0.4565	0.3840	0.0136	0.6457

Table 4. Monte Carlo study comparing mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond prices and option prices when the number of weekly observations is 2000

	ML Estimates			Jackknife Estimates		
	κ	Bond	Option	κ	Bond	Option
True Value	**0.1**	**0.8503**	**2.3921**	**0.1**	**0.8503**	**2.3921**
Mean	0.2114	0.8420	1.6784	0.0940	0.8484	2.0790
Std Dev	0.1298	0.0098	0.7911	0.1752	0.0123	1.0560
RMSE	0.1711	0.0128	1.0655	0.1753	0.0125	1.1014
True Value	**0.15**	**0.8458**	**1.9955**	**0.15**	**0.8458**	**1.9955**
Mean	0.2610	0.8384	1.3535	0.1461	0.8439	1.6787
Std Dev	0.1333	0.0091	0.7441	0.1768	0.0114	1.0022
RMSE	0.1735	0.0118	0.9827	0.1769	0.0116	1.0510
True Value	**0.2**	**0.8418**	**1.6315**	**0.2**	**0.8418**	**1.6315**
Mean	0.3073	0.8352	1.0743	0.1942	0.8401	1.3340
Std Dev	0.1386	0.0086	0.6841	0.1793	0.0109	0.9303
RMSE	0.1753	0.0108	0.8823	0.1794	0.0110	0.9768
True Value	**0.25**	**0.8381**	**1.2918**	**0.25**	**0.8381**	**1.2918**
Mean	0.3545	0.8320	0.8285	0.2370	0.8367	1.0259
Std Dev	0.1511	0.0086	0.6267	0.1967	0.0109	0.8617
RMSE	0.1837	0.0105	0.7794	0.1972	0.0110	0.9018
True Value	**0.3**	**0.8348**	**0.9735**	**0.3**	**0.8348**	**0.9735**
Mean	0.4030	0.8294	0.6292	0.2889	0.8335	0.7691
Std Dev	0.1571	0.0081	0.5380	0.1992	0.0103	0.7439
RMSE	0.1878	0.0097	0.6388	0.1995	0.0103	0.7715
True Value	**0.4**	**0.8290**	**0.4370**	**0.4**	**0.8290**	**0.4370**
Mean	0.5004	0.8248	0.3283	0.3854	0.8281	0.3761
Std Dev	0.1748	0.0075	0.3940	0.2103	0.0093	0.5421
RMSE	0.2016	0.0086	0.4087	0.2109	0.0093	0.5455

Table 5. Monte Carlo estimates of mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond prices and option prices with $m = 4$ when the true value of $\kappa = 0.1$.

	Parameter		κ	Bond Price	Option Price
	True Value		**0.1**	**0.8503**	**2.3921**
300 Monthly Obs	ML	Mean	0.2762	0.8381	1.4071
		Std Dev	0.1853	0.0125	0.8959
		RMSE	0.2557	0.0174	1.3315
	Jackknife	Mean	0.0845	0.8443	1.6636
		Std Dev	0.2179	0.0138	1.0663
		RMSE	0.2184	0.0151	1.2914
600 Monthly Obs	ML	Mean	0.1845	0.8437	1.8085
		Std Dev	0.1013	0.0080	0.6920
		RMSE	0.1319	0.0103	0.9052
	Jackknife	Mean	0.0901	0.8483	2.0831
		Std Dev	0.1156	0.0089	0.8003
		RMSE	0.1160	0.0091	0.8579
1000 Weekly Obs	ML	Mean	0.3270	0.8357	1.2827
		Std Dev	0.2371	0.0144	0.9322
		RMSE	0.3282	0.0205	1.4491
	Jackknife	Mean	0.0786	0.8424	1.5315
		Std Dev	0.2796	0.0160	1.1201
		RMSE	0.2804	0.0179	1.4126
2000 Weekly Obs	ML	Mean	0.2114	0.8420	1.6784
		Std Dev	0.1298	0.0098	0.7911
		RMSE	0.1711	0.0128	1.0655
	Jackknife	Mean	0.0905	0.8473	1.9556
		Std Dev	0.1511	0.0109	0.9265
		RMSE	0.1514	0.0113	1.0242

Table 6. Monte Carlo estimates of mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond price and option price for the (true) Vasicek model using a (misspecified) fitted CIR model. "Strike" is the ratio of the strike price to the present value of the principal and the sample size is 600.

	Parameter		κ	Bond Price	Option Price
Strike	**True Value**		**0.1**	**0.8371**	**6.1961**
0.95	ML of CIR	Mean	0.1797	0.8226	4.8922
		Std Dev	0.0940	0.0188	1.6754
		RMSE	0.1232	0.0238	2.1230
	Jackknife (m=2)	Mean	0.0892	0.8348	5.8313
		Std Dev	0.1369	0.0250	2.2311
		RMSE	0.1373	0.0251	2.2607
	Jackknife (m=4)	Mean	0.0855	0.8328	5.5596
		Std Dev	0.1106	0.0208	1.9211
		RMSE	0.1116	0.0212	2.0238
	ML of Vasicek	Mean	0.1890	0.8217	4.8858
		Std Dev	0.1026	0.0202	1.6782
		RMSE	0.1358	0.0255	2.1291

Table 7. Monte Carlo estimates of mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond price and option price for the (true) Vasicek model using a (misspecified) fitted CIR model. "Strike" is the ratio of the strike price to the present value of the principal and the sample size is 600.

	Parameter		κ	Bond Price	Option Price
Strike	**True Value**		**0.1**	**0.8371**	**2.2974**
1	ML of CIR	Mean	0.1796	0.8226	1.3780
		Std Dev	0.0940	0.0188	1.0482
		RMSE	0.1232	0.0237	1.3901
	Jackknife (m=2)	Mean	0.0889	0.8348	1.7474
		Std Dev	0.1368	0.0250	1.4649
		RMSE	0.1373	0.0251	1.5625
	Jackknife (m=4)	Mean	0.0856	0.8328	1.6071
		Std Dev	0.1105	0.0208	1.2325
		RMSE	0.1115	0.0212	1.4095
	ML of Vasicek	Mean	0.1890	0.8217	1.4668
		Std Dev	0.1026	0.0202	1.0300
		RMSE	0.1358	0.0255	1.3231

Table 8. Monte Carlo estimates of mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond price and option price for the (true) Vasicek model using a (misspecified) fitted CIR model. "Strike" is the ratio of the strike price to the present value of the principal and the sample size is 600.

	Parameter		κ	Bond Price	Option Price
Strike	**True Value**		**0.1**	**0.8371**	**0.2217**
1.05	ML of CIR	Mean	0.1797	0.8226	0.0939
		Std Dev	0.0940	0.0188	0.1643
		RMSE	0.1232	0.0237	0.2081
	Jackknife (m=2)	Mean	0.0894	0.8348	0.1159
		Std Dev	0.1370	0.0250	0.2289
		RMSE	0.1374	0.0251	0.2522
	Jackknife (m=4)	Mean	0.0853	0.8328	0.1101
		Std Dev	0.1106	0.0208	0.1973
		RMSE	0.1111	0.0212	0.2267
	ML of Vasicek	Mean	0.1890	0.8217	0.1326
		Std Dev	0.1026	0.0202	0.1877
		RMSE	0.1358	0.0255	0.2078

Table 9. Monte Carlo estimates of mean, standard deviation, RMSE of ML and jackknife estimates of κ, bond prices and option prices of ML, jackknife with $m = 4$ and MUE.

	Parameter		κ	Bond Price	Option Price
	True Value		**0.1**	**0.8503**	**2.3921**
300 Monthly Obs	ML	Mean	0.2762	0.8381	1.4071
		Std Dev	0.1853	0.0125	0.8959
		RMSE	0.2557	0.0174	1.3315
	Jackknife ($m = 2$)	Mean	0.0988	0.8460	1.8186
		Std Dev	0.2606	0.0163	1.2555
		RMSE	0.2606	0.0169	1.3803
	Jackknife ($m = 4$)	Mean	0.0845	0.8443	1.6636
		Std Dev	0.2179	0.0138	1.0663
		RMSE	0.2184	0.0151	1.2914
	MUE	Mean	0.0916	0.8554	2.8015
		Std Dev	0.1953	0.0168	1.4050
		RMSE	0.1954	0.0173	1.4634
	Jackknife Plug-in ($m = 2$)	Mean	0.0988	0.8584	2.8825
		Std Dev	0.2606	0.0564	2.1560
		RMSE	0.2606	0.0570	2.2110
	Jackknife Plug-in ($m = 4$)	Mean	0.0845	0.8562	2.9417
		Std Dev	0.2179	0.0223	1.7644
		RMSE	0.2184	0.0231	1.8480

Table 10. Summary statistics and unit root tests for monthly Fed fund rate from July 1954 to June 2002 and weekly Fed fund rate from July 7, 1954 to December 21, 1994.

Data Frequency	Monthly	Weekly
Number of Observations	576	2112
Mean	0.0600	0.0617
Standard Deviation	0.0333	0.0357
Autocorrelation ρ_1	0.9815	0.9943
Autocorrelation ρ_2	0.9521	0.9897
Autocorrelation ρ_3	0.9226	0.9839
Autocorrelation ρ_4	0.8954	0.9789
Autocorrelation ρ_5	0.8721	0.9719
Autocorrelation ρ_6	0.8515	0.9657
Autocorrelation ρ_7	0.8330	0.9586
$Z(t)$ test	-2.5925	-2.6386
5% critical value	-2.8669	-2.8634
10% critical value	-2.5696	-2.5678

Table 11. Empirical estimates for monthly Fed fund rate from July 1954 to June 2002.

Method	κ	Bond Price	Option Price			
			Option Expiration (t:years)	$\frac{\text{Exercise Price}}{\text{Current Val of Par}}$		
				.95	1	1.05
ML	0.1018	0.8408	0.5	7.1902	3.6398	1.1318
Jackknife	0.0846	0.8434		7.4402	3.8677	1.2766
ML			1	9.3080	5.4600	2.1340
Jackknife				9.5386	5.6842	2.3148

Table 12. Empirical estimates for weekly Fed fund rate from July 7, 1954 to December 21, 1994.

Method	κ	Bond Price	Option Price			
			Option Expiration (t:years)	$\frac{\text{Exercise Price}}{\text{Current Val of Par}}$		
				.95	1	1.05
ML	0.3550	0.8386	0.5	7.0425	3.5520	1.0502
Jackknife	0.2222	0.8388		7.1613	3.7538	1.2696
ML			1	9.1322	5.3706	2.1700
Jackknife				9.2021	5.4934	2.3281

Figure 1: Distribution of jackknife and ML estimates of bond option prices based on 600 monthly observations.

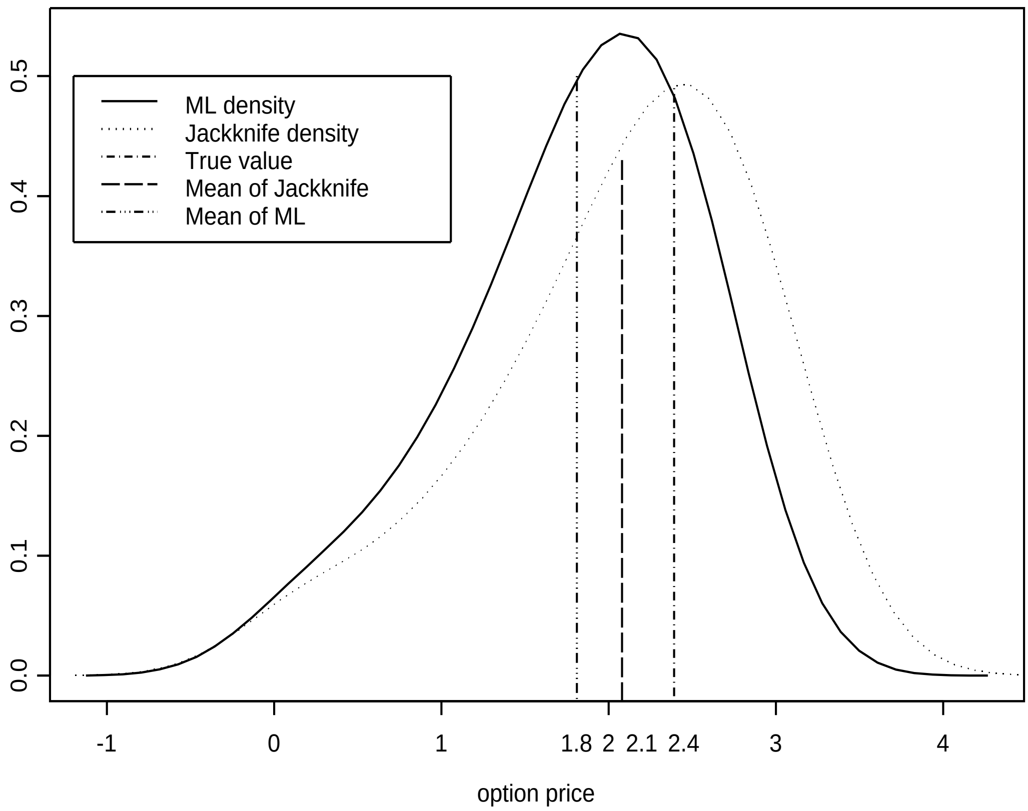

Figure 2: Relationship between κ and option and bond prices.

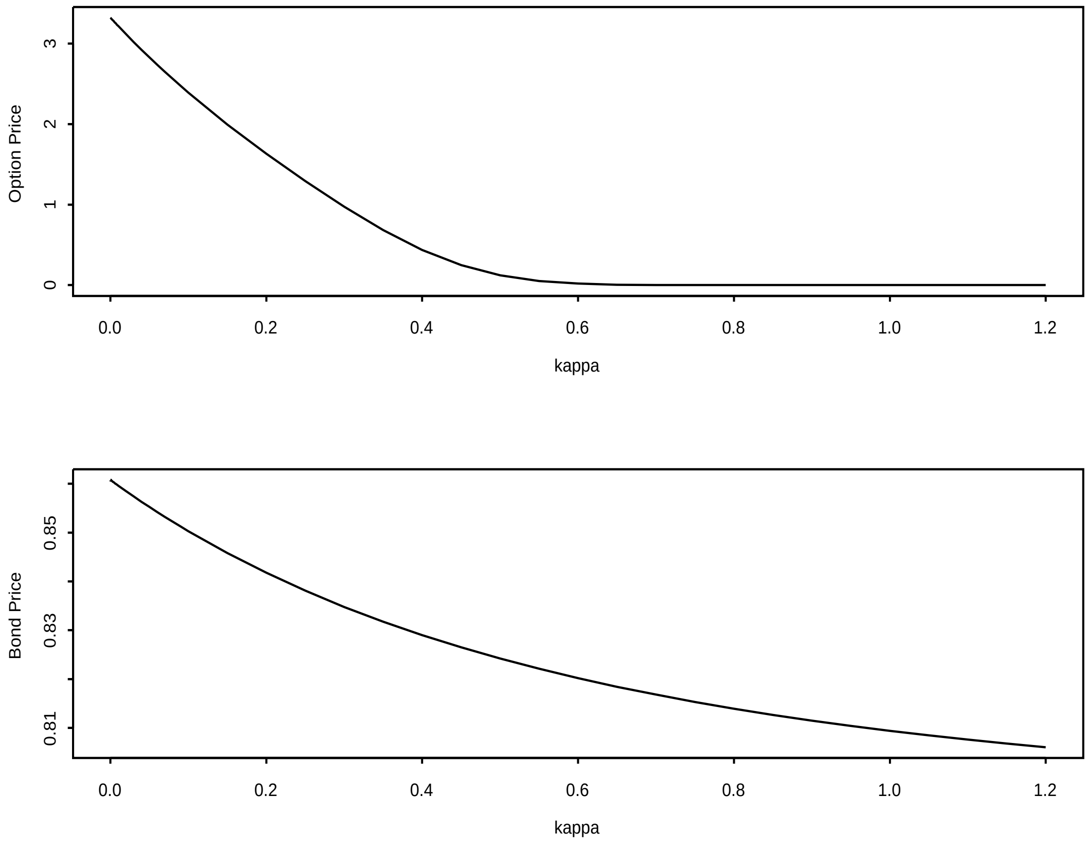

Figure 3: Percentage bias in κ, bond price, and bond option price based on ML estimation graphed as a function of κ. Sample size = 300, sampling frequency = monthlyly.

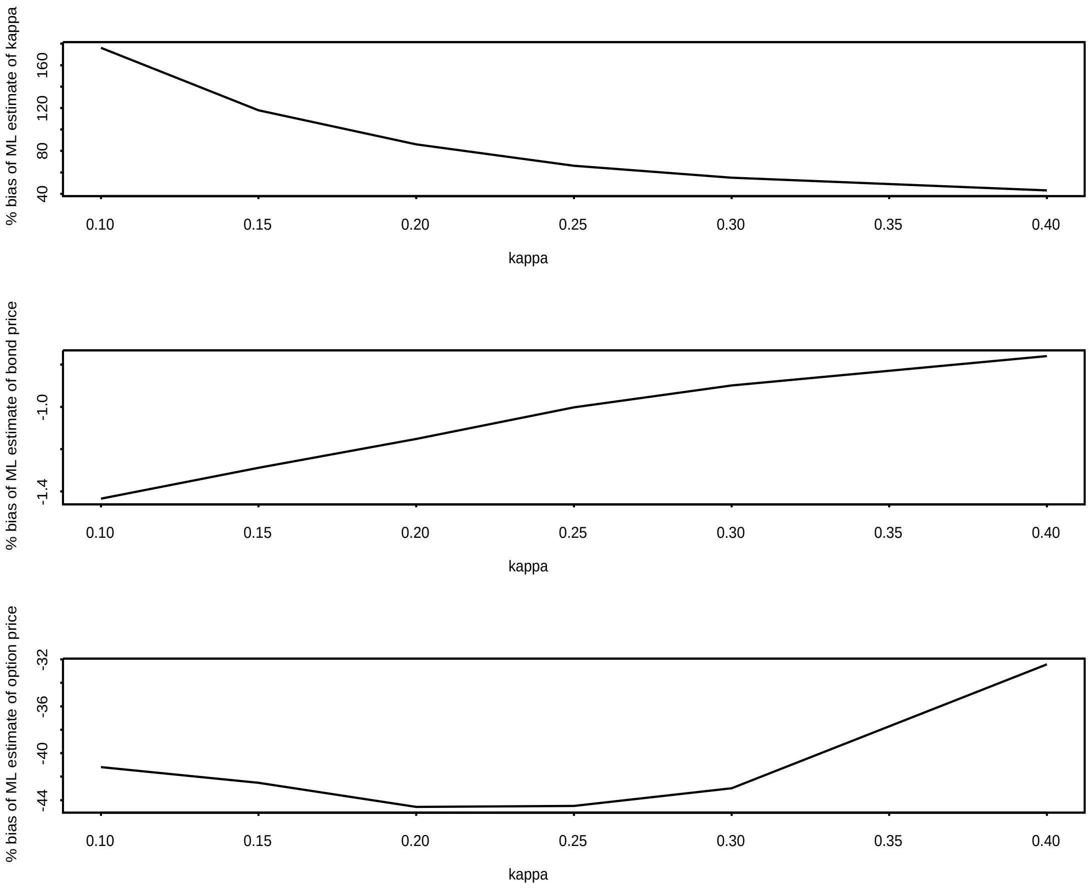

Figure 4: Percentage bias in κ, bond price, and bond option price based on ML estimation graphed as a function of κ. Sample size = 600, sampling frequency = monthly.

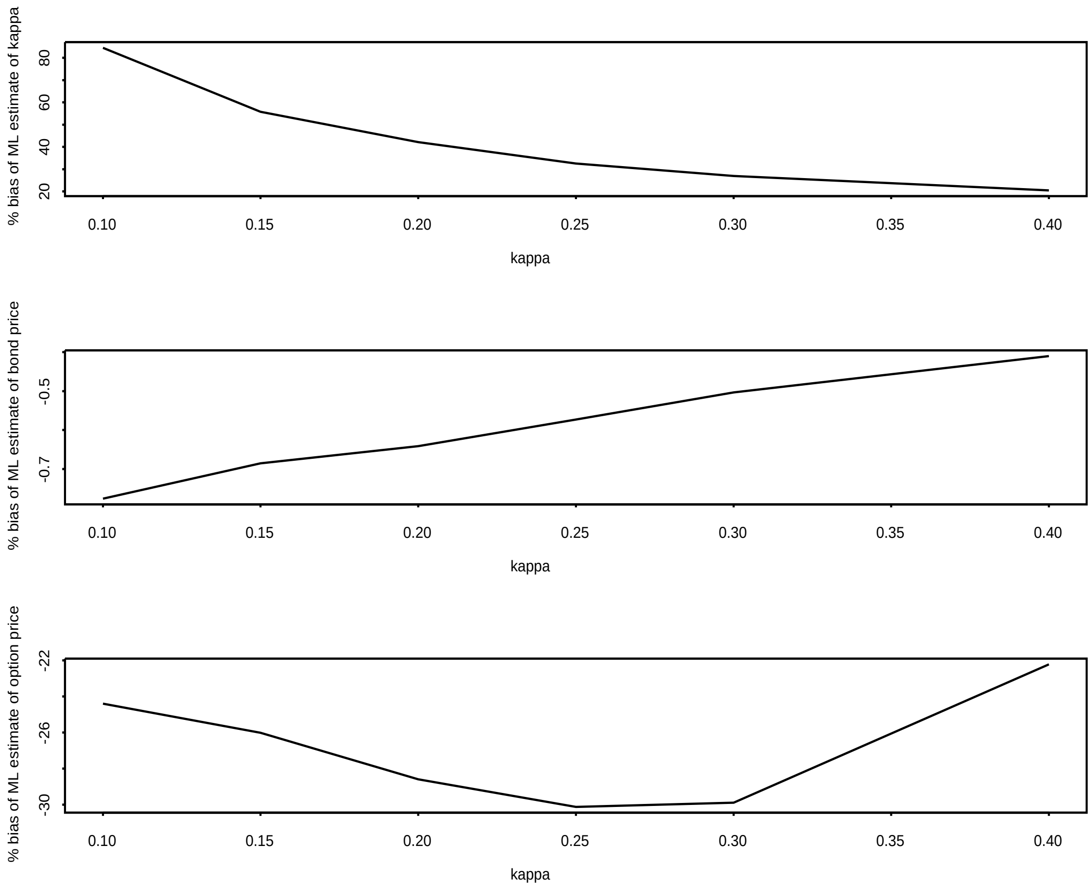

Figure 5: Percentage bias in κ, bond price, and bond option price based on ML estimation graphed as a function of κ. Sample size = 1000, sampling frequency = weekly.

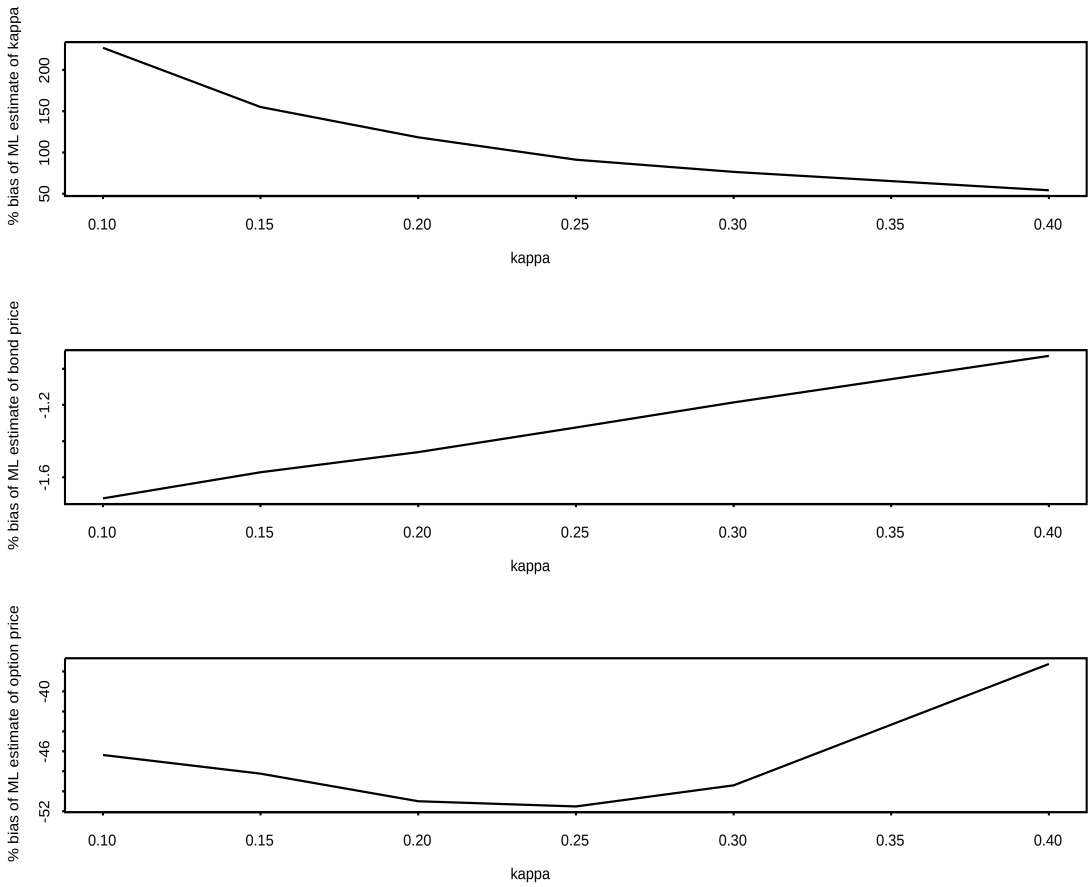

Figure 6: Percentage bias in κ, bond price, and bond option price based on ML estimation graphed as a function of κ. Sample size = 2000, sampling frequency = weekly.

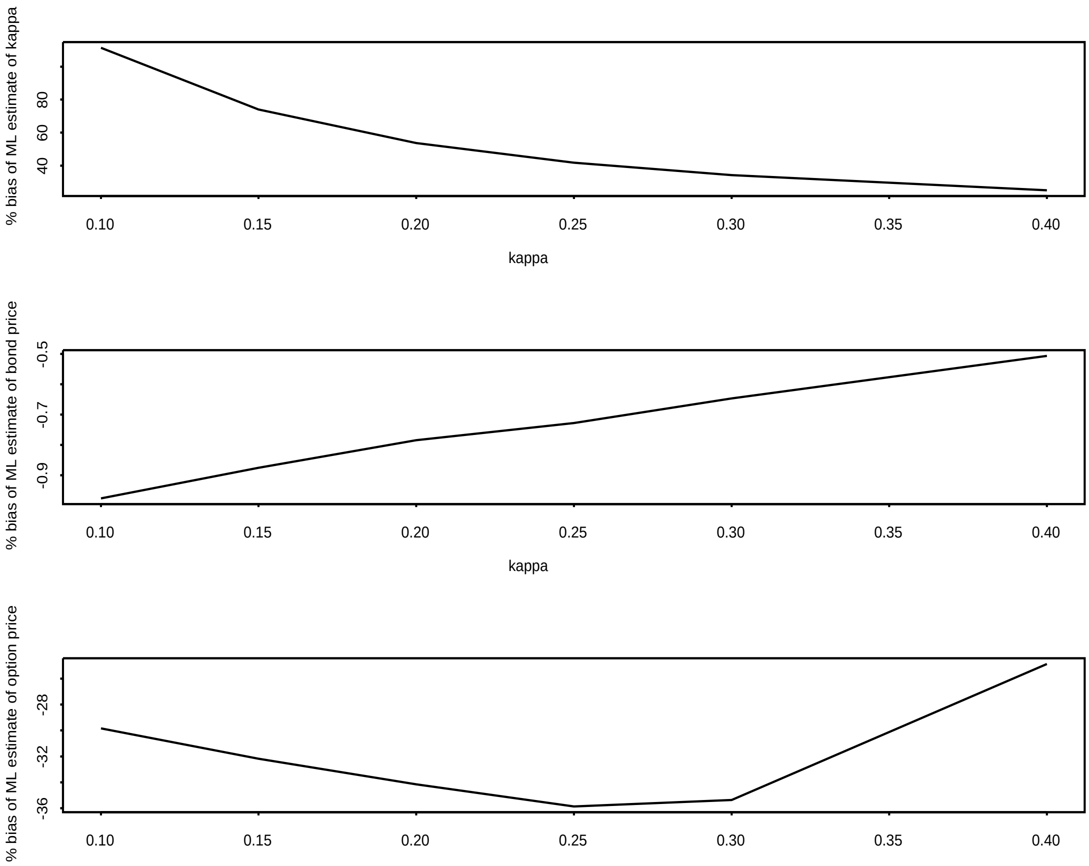

Figure 7: Percentage bias in κ, bond price, and bond option price graphed as a function of κ. Sample size = 300 and sampling frequency = monthly. Marked line = jackknife, solid line = MLE.

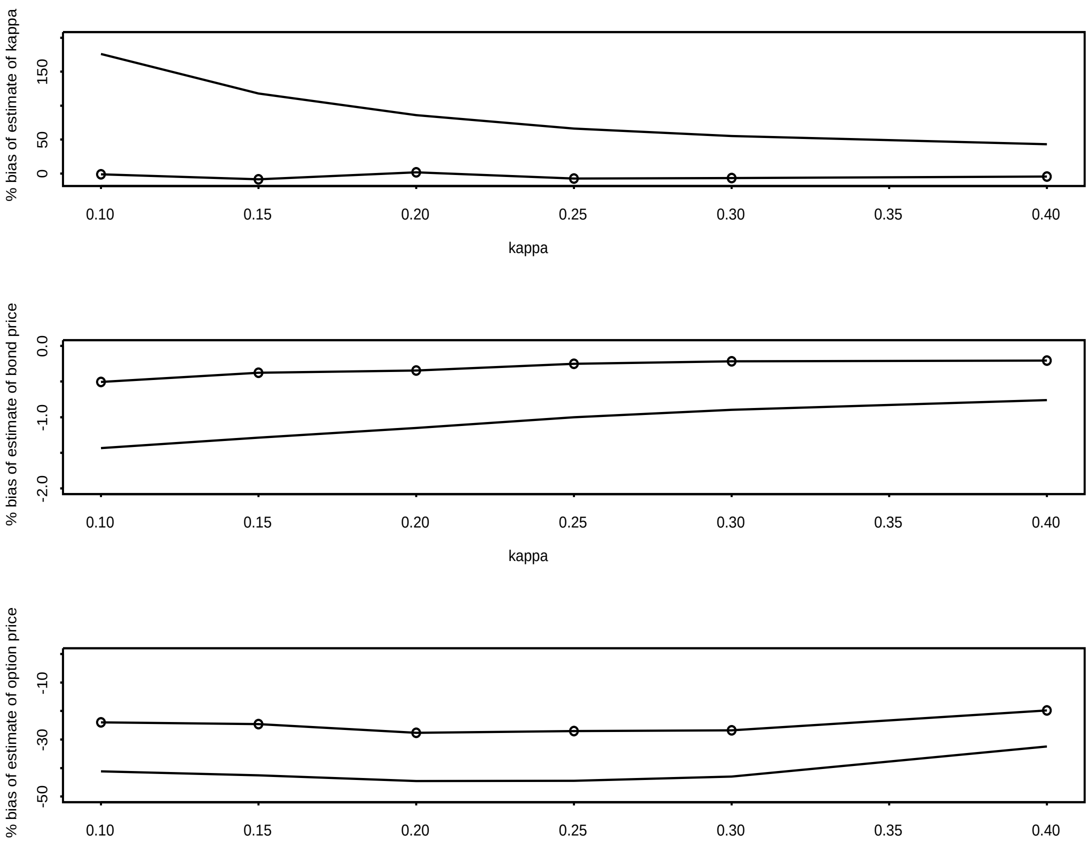

Figure 8: Percentage bias in κ, bond price, and bond option price graphed as a function of κ. Sample size = 600 and sampling frequency = monthly. Marked line = jackknife, solid line = MLE.

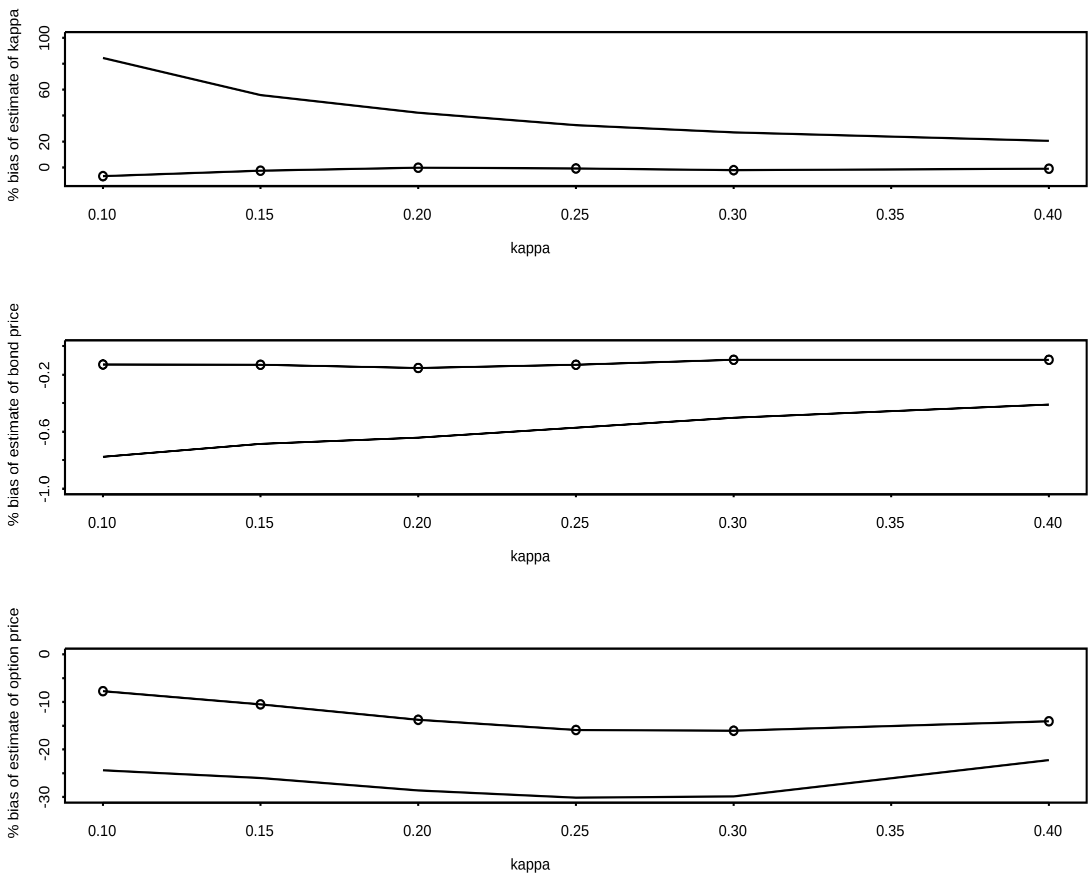

Figure 9: Percentage bias in κ, bond price, and bond option price graphed as a function of κ. Sample size = 1000 and sampling frequency = weekly. Marked line = jackknife, solid line = MLE.

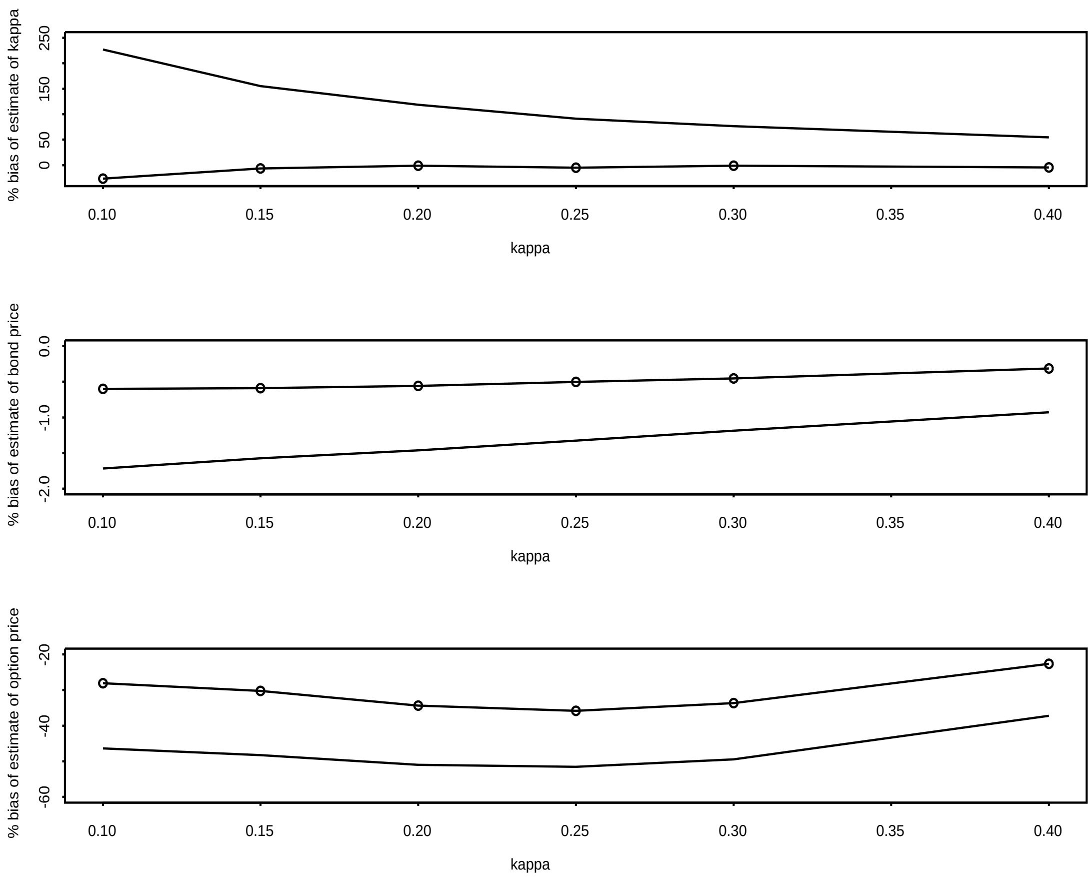

Figure 10: Percentage bias in κ, bond price, and bond option price graphed as a function of κ. Sample size = 2000 and sampling frequency = weekly. Marked line = jackknife, solid line = MLE.

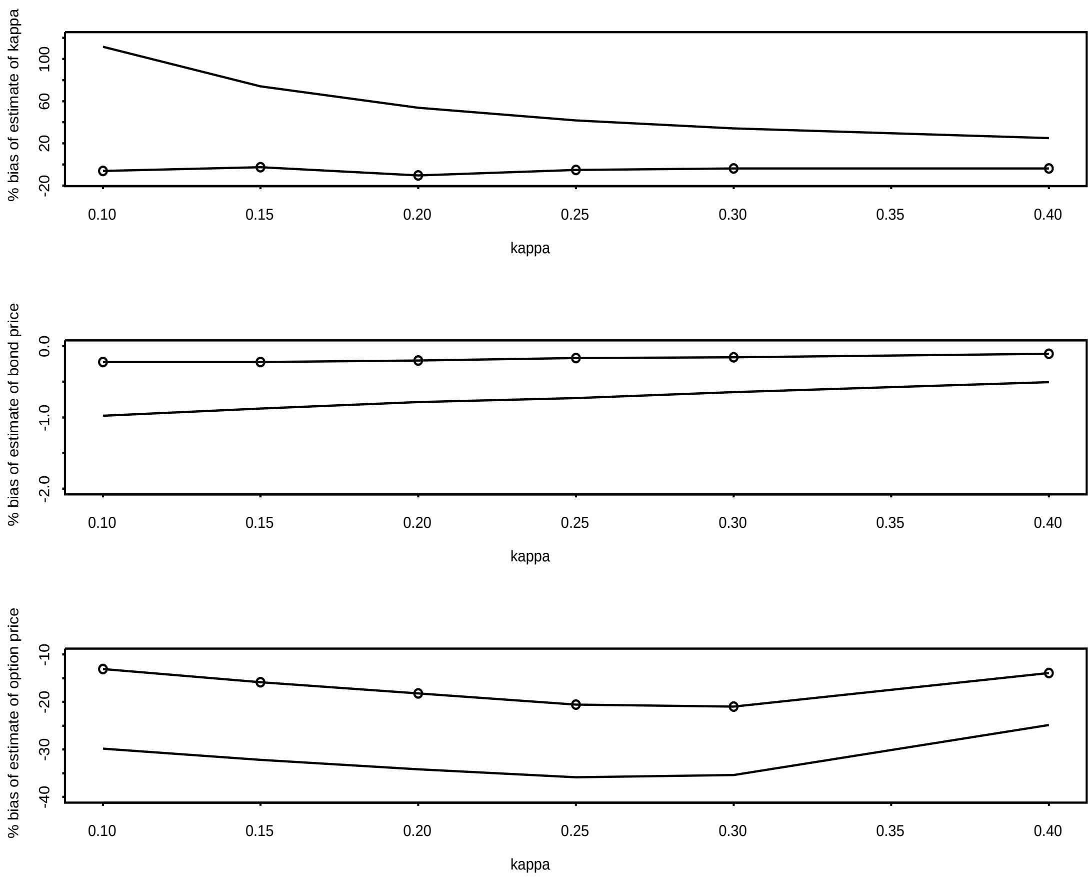

Figure 11: Density of ML estimate and jackknife estimate (m=4) of kappa based on 300 monthly observations

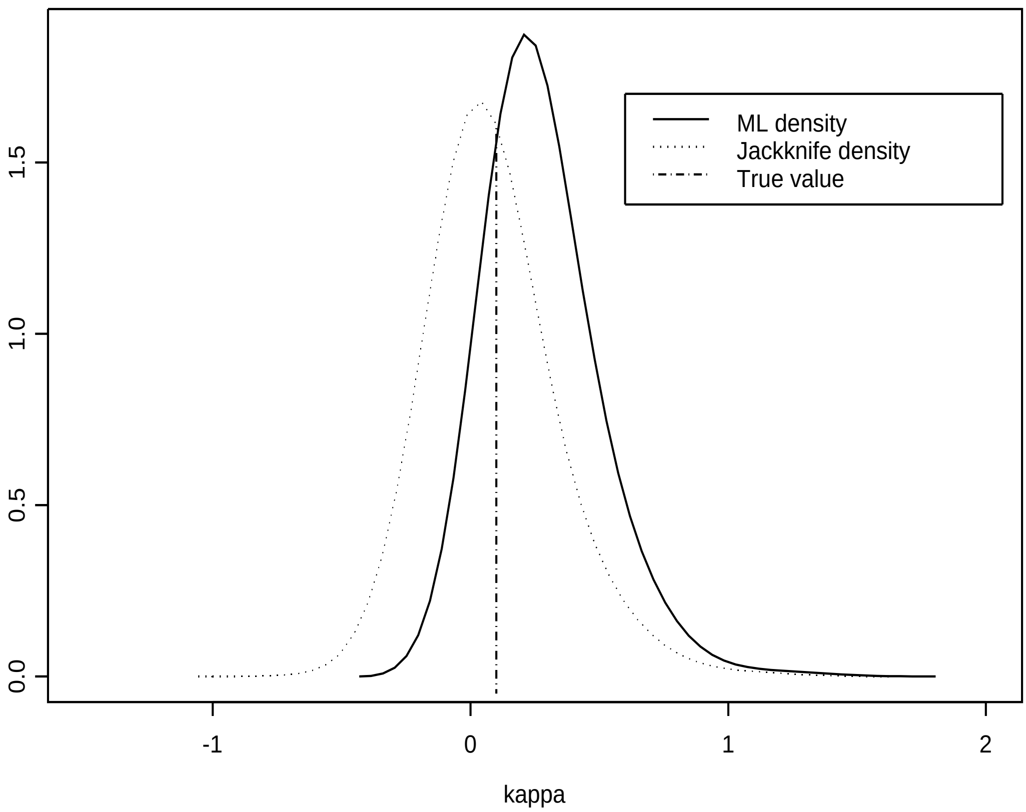

Figure 12: Densities of ML estimate, jackknife estimate (m=4) and MUE of kappa based on 300 monthly observations.

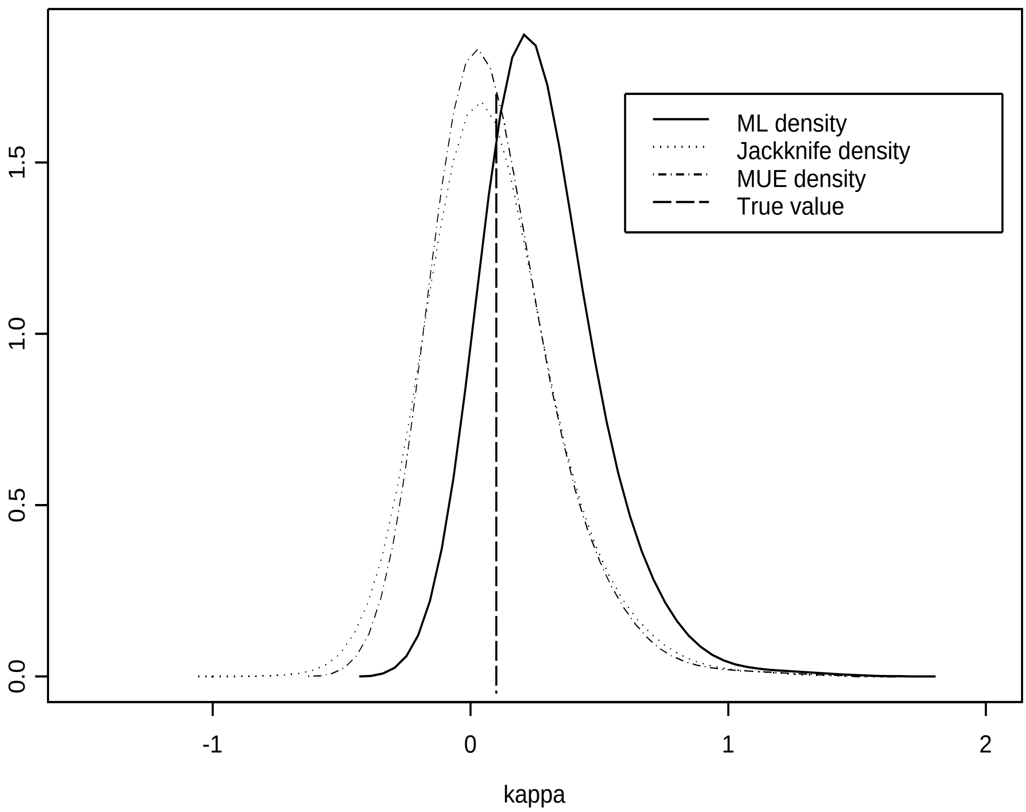

Figure 13: Time series plots of monthly and weekly Fed funds rate