

**Valuing Real Options using Implied Binomial Trees
and Commodity Futures Options**

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Abstract:

We show how to value a real option on a commodity using an implied binomial tree (IBT) that is calibrated using commodity futures options prices. Until now it has been assumed that spot options are required to be traded on the underlying asset in order to use an IBT; this requirement is, however, typically not met with commodities. We make two major contributions: First, by showing how to implement an IBT when no directly traded options exist, we open the door to calibration of real option models to market-implied probability distributions for commodity prices—including excess skewness and kurtosis, and perceived likelihood of jumps. This is a significant relaxation of the traditional binomial tree's rigid assumption that commodity prices be distributed lognormal. Second, the existence of long-dated futures options means that our technique allows good volatility estimates to now be incorporated into capital budgeting evaluations of real options projects with long planning horizons. We give an example using gold futures options and a real option to extract gold from a mine.

I. Introduction

We show how to use implied binomial trees (IBTs) for the valuation of real options where the underlying asset that drives the value of the real option is a physical commodity for which futures options exist but for which spot options do not exist (e.g., gold, silver, platinum, oil, wheat). The absence of spot options is the norm for commodities, and this means that until now it has not been possible to calibrate an IBT of commodity prices using traded options. By showing how to fit an IBT to commodity prices without using spot options, we enable the estimation of binomial option pricing models that completely relax the traditional binomial tree's rigid assumption that commodity prices be distributed lognormal.¹ In addition, the existence of long-dated futures options means that good volatility estimates may now be incorporated into capital budgeting evaluations of real options projects with long planning horizons.² We give an example using a gold mine, but the technique also applies to non-investment commodities.

II. Implied Binomial Trees

Implied binomial trees (IBT) infer consensus market views on future risk-neutral distributions for the underlying asset. By calibrating the tree to the market prices of traded options, skewness and kurtosis within the distribution of returns to the underlying asset emerge in the tree and can be used to price other, possibly non-traded, options based

¹ To be precise, the lognormal assumption is true only in the limit as step size goes to zero in a traditional binomial tree. For non-infinitesimal time steps, the distribution of asset price is only approximately lognormal in the traditional binomial tree.

² One other benefit of IBTs that is not often pointed out is that, by construction, the local volatility of the underlying is allowed to vary through the binomial tree (Rubinstein [1994, p795]). This is another relaxation of the traditional binomial tree's assumptions.

on the tree. Some of these higher moments implicitly account for jump probabilities although not modelled explicitly as such. The IBT produces these benefits with much less quantitative machinery than many other analytic techniques.

The three best-known implied binomial tree techniques are due to Derman and Kani (1994), Rubinstein (1994), and Jackwerth (1997). Each technique has different strengths and weaknesses. The core differences between techniques that are relevant for the present paper are:

- Jackwerth's 1997 generalized implied binomial tree (J-GIBT) is calibrated using American-style options of different maturities, whereas Rubinstein's 1994 implied binomial tree (R-IBT) is calibrated using only European-style options of a single maturity.
- The R-IBT is quite inflexible with respect to the internal probability structure of the binomial tree. The J-GIBT, however, allows for more flexibility so that intermediate maturity and American-style options may be priced.
- R-IBT requires either all calls or all puts, but J-GIBT does not.

III. Literature on Pricing Commodity Futures Options and Commodity Futures

The most important factors affecting commodity prices are general shifts in the cost of producing the commodity and general shifts in the demand for the commodity (Black [1976], p. 174). Interest rates and storage costs are important to a lesser extent. Changes in demand and supply are reflected in the convenience yield.

Let δ denote the average continuously-compounded net convenience yield (consumption benefit less storage costs) for a commodity from time- t (now) to time- T (expiration of a futures contract on the commodity). Let r denote the average continuously-compounded riskless interest rate over the same period. Let S denote the spot price of the commodity at time- t . Then a fair price $F(t, T)$ at time- t for future delivery of the commodity at time- T is given by $F(t, T) = S_t e^{(r-\delta)(T-t)}$. In the case of a physical commodity, δ is typically not directly observable, but it may be deduced.³

The consumption benefit is the benefit of direct access to a commodity. The owner of the commodity (possibly the short position on a futures contract) receives the consumption benefit, but the owner of the futures on the same commodity does not receive the consumption benefit. The classic case is a Kellogg Cornflake factory with a silo of corn waiting to go into production; even if the futures price is very low relative to the spot, Kellogg will not sell the corn in exchange for a long futures contract because they cannot make cornflakes out of futures contracts. Thus, high convenience yields imply low futures prices, other things being equal.

The practical implications are slightly different for gold and silver and other investment commodities held by many people for investment motives. A very low futures price relative to the spot leads investors to sell physical gold in exchange for a long futures contract. The marginal trader is not holding gold and silver for consumption, so the marginal consumption benefit is low.

In the case of a financial security, there is no storage cost and δ is just the continuous dividend yield. Indeed, Carmona and Ludkovski (2003) consider the

³ The average δ may be deduced from futures and spot prices, and the instantaneous δ may be able to be deduced from, for example, gold lease rates or from a term structure of δ inferred from futures prices.

convenience yield as a correction to the drift for the spot process that is in essence a dividend. The main concern in this investigation is in how the convenience yield may change through time.

The interaction of spot, futures, and convenience yield is complicated. Gibson and Schwartz (1990) present an analytical two-factor model for oil prices (the two factors are randomness in the price of oil and in the instantaneous convenience yield). The authors estimate the market price of convenience yield risk and find it to be negative (suggesting that it pays to bear convenience yield risk). A forward curve of convenience yields is implied by forward contracts of different maturity. The curve displays considerable mean reversion in convenience yields—driven by oil inventories, shortages, fears of Middle East wars etc.

Miltersen and Schwartz (1998) present analytical models for pricing commodity futures options. The models allow for stochastic convenience yields and stochastic interest rates, and are a closed-form generalization of the Black-Scholes formula. Hilliard and Reis (1998) attack a similar problem to Miltersen and Schwartz but their model allows for jump diffusions in commodity prices and for excess (positive or negative) skewness in the distribution of the commodity price. The Hilliard-Reis model also allows for a no-arbitrage term structure of convenience yields and is a three-factor model (spot, convenience yields, and jumps or alternatively spot, convenience yields and interest rates).

Hilliard and Reis (1998) present a real option valuation problem (the price of one barrel of oil) using their forward pricing formula. They provide a good discussion of why assuming that an instantaneous convenience yield remaining at its current level rather

than reverting to a long run mean can bias pricing (Hilliard and Reis [1998], pp. 78–79). Further, a positive correlation between the spot price process and the spot convenience yield generates mean reverting behavior to the spot commodity price (Miltersen and Schwartz [1998], p. 34).

IV. Properties of our Method

To implement the method, choose a commodity and a futures contract on that commodity maturing at time T . Assume that the net convenience yield δ and the riskless interest rate r between time- t and time- T are non-stochastic and equal to the convenience yield implied by the futures contract and the riskless rate implied by Treasuries of this maturity, respectively. This implies that stochastic changes in futures prices are due solely to stochastic changes in the underlying commodity price (a one-factor model).

For this investigation, an implied binomial tree for gold futures prices is inferred from gold futures options. No-arbitrage relations between gold futures prices and physical gold prices result in an implied binomial tree for physical gold spot prices. This innovation enables the valuation of real option projects where the value of the project is a function of the commodity price when there are no options traded directly on the commodity itself.

The method is limited by the horizon of the futures options that exist. In the case of gold, for example, contracts exist only out to two years. In the case of NYMEX light sweet crude oil, contracts exist out to 84 months. The NYMEX publishes daily fact sheets on its website that list futures settlement prices and futures options settlement prices that cover all contracts and extend well beyond the horizon of active trade. What is notable is

that futures options tend to have much longer maturities than many other option classes making them more suitable for the analysis of longer-range capital budgeting projects.

The average convenience yield implied by the futures price allows for anticipated reversion to the mean before maturity, but it ignores any conditional behavior. For example, if instantaneous convenience yield is high at time- t and expected to revert to a long-run mean, then our δ will be too low in the early nodes of the tree and too high in the latter nodes, though correct on average. This could lead to pricing biases for options with odd distributions of payoff. The technique could be directly extended, however, using a conditional modelling of the convenience yield that allows the forecasting of mean reversion in the instantaneous convenience yield over the life of the project.⁴ Plots of historical convenience yields, in the form of gold lease rates are available from [kitco.com](http://www.kitco.com).

Under these assumptions, a tree for the spot price process is inferred directly from the J-GIBT for the futures prices using the non-stochastic convenience yield and interest rate. This technique takes out the basis (i.e., difference between futures and spot) at a constant continuously-compounded rate.

Trigeorgis (1993) catalogues many types of real options: e.g., option to abandon, defer, expand, contract. These examples have the project itself as the underlying security, rather than the asset driving the value of the project. For example, when evaluating the purchase/leasing of a gold mine, the gold mine is the underlying asset, not the gold itself. Our technique differs in that a tree is generated for physical gold. However, this is still

⁴ Perhaps using futures contracts of different maturity to infer a term structure of convenience yield, and then using that term structure to infer convenience yields for different intervals in the future (like inferring forward interest rates from a term structure of spot rates).

appropriate if the value of the project is a function of the value of an underlying commodity.

V. A Real Option on a Gold Mine

Assume that on May 19, 2004, a gold mining company is offered the opportunity to bid for a 60-day right to purchase a small piece of land for which surveys estimate a yield of 4,500 troy ounces of gold. The right can be exercised on day-60, and the mining operation takes place from day-61 to day-100, at an expenditure of \$2,000,000 up front (day 60) for the land and equipment and a payment of \$500,000 up front into an escrow account to cover wages over the next 40 days. Money can be borrowed and paid back with interest at 5% per annum on day-100 ($5\% \times 40/365$ times borrowing). The operation yields 4,500 ounces of gold to sell into the spot market on day-100 (i.e. collected at a rate of 125 ounces per day), at which point the operation is abandoned and the machinery sold for \$750,000. Consequently, on day 60, a decision to proceed is made if the project value is positive. The issue becomes: what is the value of this right? To make a thorough valuation, different values of mine yields (4,000, 4,500, and 5,000 ounces) are analyzed to provide a sensitivity analysis.

VI. Initial Details of the Method

When using an implied binomial tree, traded options that have the same underlying asset as the option being valued are desired. Unfortunately, there are no options for physical gold traded on organized exchanges. However, gold futures options are traded. In this case, the underlying is a gold futures contract. The same is true for

many commodities: there are no options on the commodity itself, but there are options on futures on the commodity.⁵ There are two reasons for this: first, it is physically and administratively easier to deliver a futures contract on a commodity than to deliver the commodity; and second, the transactions costs for delivery of a futures contract on a commodity are much lower than for delivery of the commodity.

The technique developed in this paper allows one to infer an implied binomial tree for spot gold using gold futures options. The procedure is as follows (the next section discusses the implications of the institutional details of the gold futures options for the implementation of the method):

1. Identify a futures contract on gold with maturity in excess of the real option horizon.
2. Use gold futures options to infer a Jackwerth GIBT (J-GIBT) for futures prices out to the maturity of the futures contract (the options are American style, making the J-GIBT model the better choice over the R-IBT model). The options expire one month before the futures in this example. To initiate the J-GIBT estimation, build a CRR tree for the futures prices and focus only on the final nodal futures values as inputs to the J-GIBT. Using three different objective functions (with corresponding constraints) for the J-GIBT (see details in the next section), final nodal probabilities are found and then an estimated generalized weight function is used to retrieve nodal probabilities backward throughout the tree (i.e. an implied binomial tree).

⁵ For example, the NYMEX trades futures options on light sweet crude oil, natural gas, heating oil, gasoline, Brent crude oil, propane, coal, electricity, gold, silver, copper, aluminum, platinum, and palladium; the CBOT trades futures on corn, soybeans, wheat, oats, rice, gold and silver; and the CME trades futures options on beef, dairy, hogs and lumber. The LIFFE trades futures options on cocoa, coffee, wheat, and white sugar.

3. Assert that by no arbitrage, the values and risk-neutral nodal probabilities for the final nodes of the J-GIBT for gold futures prices carry over to the values and risk-neutral nodal probabilities for a tree of *physical* gold spot prices. This is appropriate because the futures price and the spot price must be identical at the maturity of the gold futures contract (Hull [2000, p36]). That is, although one may not know what the futures price will be at maturity of the futures contract, it is known for certain that the futures price is equal to the spot price at maturity ($S_T = F_T$), and therefore the final nodes for the futures tree must be identical to the final nodes for a spot tree.
4. Assuming a constant convenience yield, a spot price tree that captures the J-GIBT weight function can be inferred from the futures price tree. Simply assume at time- t' during the tree, that $F(t', T) = S_{t'} e^{(r-\delta)(T-t')}$ and deduce $S_{t'} = F(t', T) e^{-(r-\delta)(T-t')}$ where δ is the average convenience yield initially inferred from the current spot and futures prices.^{6,7}
5. Value the real option on physical gold using the IBT for the gold spot process.

⁶ An alternative is to ignore the J-GIBT generalized weight function after it is estimated and build an R-IBT tree (Rubinstein [1994, p790]) backwards from the ending nodal values of the J-GIBT. These two techniques give different answers because the distribution of probability through each tree is different (they do agree, however, if the J-GIBT weights are linear). We do this but do not report the R-IBT results.

⁷ Another alternative is to think entirely in terms of futures prices. We could then create an implied binomial tree for futures prices using the J-GIBT method and the American-style futures options, and estimate the gain or loss on our futures contract if we physically deliver gold. This restricts you to valuing real options where you will in fact make or take delivery of the underlying using futures.

VII. Institutional Details of our Method Particular to Gold Futures Options

Gold futures and gold futures options (i.e., options with one gold futures contract as the underlying) trade on the COMEX division of the NYMEX and also on the CBOT and the CME. The COMEX gold contracts are used in this paper.

On May 19, 2004, August 2004 gold futures settled at \$376.90 per ounce. August 2004 gold futures mature 100 days later on August 27, 2004. There are also August 2004 gold futures options. These options are American style and mature on July 27, 2004 (69-day options expiring one month before the futures contract expires).⁸ The futures cover 100 troy ounces of gold. The option covers one futures contract. Both the options and futures settled at 1:30PM EST. This is the same time as the closing of the spot market in New York, at which time the bid and ask prices on physical gold are \$382.50–383.00 per ounce. The mid-spread value is used as the current gold spot price in the example.

Miltersen and Schwartz (1998) discuss the importance for option pricing of the time lag between expiration of the option and the underlying futures (Section VI). Their discussion is in the context of analytical models for pricing commodity futures options. They note that accounting for the time lag between expirations has a significant effect on the option valuation.

The fact that the futures expire in 100 days and the options expire in only 69 days means that the tree for futures prices in the example extends 31 days beyond the maturity of the options. This is unusual for an IBT, and not strictly necessary for fitting the traded option prices, however, the tree is extended in order to be able to deduce the spot tree. This extrapolation is not problematic because the underlying for the option is the unique futures contract that expires one month after the option expiration, it is not very distant in

⁸ This one-month lag is typical for commodity futures options.

time from the option expiration, and the current futures price is available for that maturity. However, as demonstrated in the results section, the extrapolation creates problems for one of the objective functions. Presumably these problems are less significant when the options have longer maturities (and thus the smaller the extrapolation as a portion of the option life).

Six August 2004 gold futures options with associated strikes and settlement prices are displayed in Table 1 (displayed later in the text). The Treasuries section of the *Wall Street Journal* of May 20, 2004 provides bid and ask quotes of 1.04–1.03 for both 98-day and 105-day T-bills, respectively. These quotes imply a 100-day continuously-compounded annualized yield $r = 1.0509\%$.⁹ The analysis is simplified slightly by using this rate for the 69-day option maturity as well.

To estimate the J-GIBT, do the following:

1. Use Black's (1976) futures options model to find an implied volatility for the closest to at-the-money futures option (the 380-strike option in this example; $\sigma = 16.873\%$).
2. Build a traditional CRR binomial tree for the futures price using σ from step 1.

Make it a 100-step tree to match the maturity of the futures. The CRR risk-neutral

⁹ We linearly interpolate between the continuously-compounded yields on the T-bills of maturities bracketing the maturity of the futures. These continuous yields are calculated using the midpoints of the WSJ quoted bid-ask spreads as per Cox and Rubinstein [1985, p255]. These yields are in fact not used to construct the futures tree, but to deduce the δ and value the futures options.

ending nodal probabilities¹⁰ P_j' are the initial guess for the Jackwerth optimization that solves for ending nodal probabilities P_j and a weight function.

3. Create a weight function for the Jackwerth tree. We consider two functions for this investigation. The first is stepwise linear from 0 to 0.5 and then from 0.5 to 1, taking the values 0 at 0, α at 0.5, and 1 at 1. The second, reported here, is stepwise linear with ten sections at 0.1 intervals from 0 to 1.0. Taking the initial values $\alpha_0 = 0.00$ at 0, $\alpha_1 = 0.10$ at 0.1, $\alpha_2 = 0.20$ at 0.2, and so on up to $\alpha_9 = 0.90$ at 0.9, and $\alpha_{10} = 1.00$ at 1.0. The pricing is slightly better with the latter weight function.
4. The optimization problem for the Jackwerth tree is set up as follows:

$$\min_{\alpha_1 - \alpha_9, P_0 - P_{100}} \sum_i (C_i^{MKT} - C_i^{J-GIBT})^2$$

subject to

$\sum_j P_j = 1$ and $P_j \geq \varepsilon$ for $j = 0, \dots, n$, for small positive ε , and F equals the first node of the J-GIBT for futures prices.

Where C_i^{J-GIBT} is the option price implied by the J-GIBT tree accounting for the possibility of early exercise at each node, for $i = 1, \dots, m$, but necessarily using only the first 69 steps of the 100-step tree. This implementation is labelled “JW-OBJ” (for “Jackwerth Objective”) in tables and graphs.

¹⁰ For ending nodes numbered from 0 (lowest futures price) up to 100 (highest futures price) these probabilities are $P_j' = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$ where $p' = \frac{1-d}{u-d}$ and $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ are the growth factors. Note that the risk-neutral drift of the futures is zero.

The ending nodal probabilities and the weight function are also estimated using two more objective functions. The first of the additional objective functions is proposed by Rubinstein (Rubinstein [1994]). The sum of squared deviations of the ending nodal probabilities from those of the CRR tree for the futures (this requires that the option pricing moment conditions go into the constraints) are minimized. The CRR distribution is lognormal, but the objective function allows significant deviation from lognormality. This objective function is labelled “RUB-OBJ” in the tables and graphs. The second of the additional objective functions is a “smooth” objective function proposed by Jackwerth and Rubinstein (1996). The objective is simply: $\sum (P_{j+1} - 2P_j + P_{j-1})^2$ where $P_{-1} = P_{n+1} = 0$. Again, this requires that the option pricing moment conditions be placed into the constraints. This objective function is labelled “SM-OBJ” in the tables and graphs.

5. Following estimation of the Jackwerth tree, the spot prices for the physical gold process for a 100-day tree using $S_{t'} = F(t', T)e^{-(r-\delta)(T-t')}$ as described earlier are inferred ($\delta = -0.139\%$, a negative convenience yield).

There are several notes. *First*, J-GIBT uses a weight function to allocate nodal probability weight backwards through the tree. This is more flexible than Rubinstein’s 1994 IBT technique because it provides degrees of freedom within the body of the tree, rather than just at the nodes. It is this flexibility that permits the pricing of American-style options that may need to be exercised early (and also options of multiple maturities). *Second*, the

risk-neutral drift of the futures price in the initial CRR tree and in the J-GIBT tree is zero. The futures price is not the price of a traded asset, so the drift need not be the riskless rate.¹¹ There is riskless drift in the J-GIBT for the options of course.¹² *Third*, we use additional objective functions (RUB-OBJ and SM-OBJ) because the Jackwerth objective function produced spiky non-credible ending nodal distributions for the underlying when extrapolated out to the futures maturity.

For comparison, we produce two other binomial trees for the spot gold process. The first is an R-IBT built backwards from the ending nodes of the J-GIBT and using the initial spot price of gold.¹³ The second is a traditional CRR tree built by sampling the most recent 61 business days of physical gold prices¹⁴ and estimating the volatility of spot gold prices.¹⁵

¹¹ Think about valuing the futures contract, as opposed to finding the futures price: the value must be zero. If we assume no interest on the margin account, then the value (i.e., 0) is just $\sum P_j [F(T,T)_j - F(t,T)]$. Thus, $F(t,T) = \sum P_j F(T,T)_j$ with no discounting and the drift is zero. Alternatively, let E^* denote RN expectation, then $E^*[F(T,T)|F(t,T),S(t)]$ must equal $E^*[S(T)|F(t,T),S(t)]$ by no arbitrage at time T. But the latter is just $F(t,T)$. Thus $F(t,T) = E^*[F(T,T)|F(t,T),S(t)]$ and there is no drift.

¹² It is here that we should technically have used the Treasuries for the maturity of the options as opposed to the futures.

¹³ The J-GIBT weight function is ignored when building the Rubinstein IBT backwards. There is no Rubinstein optimization when building the Rubinstein IBT. Rather, the optimization occurs at the Jackwerth tree calibration and the ending nodal probabilities feed directly into the Rubinstein recursion. This means that the ending nodal probabilities from the Jackwerth tree are inferred assuming that path probabilities need not be equal, but the Rubinstein IBT that uses them assumes that path probabilities are equal. We do not report these results; they lie roughly within the range of results from the three objective functions used for the J-GIBT.

¹⁴ We used London AM Gold Fix data from www.kitco.com from Feb 20, 2004 to May 19, 2004.

¹⁵ We found daily standard deviation of 0.0122665400, and multiplied by square root of 252 to get an annualised number of $\sigma = 0.194725286$. We then used the 100-day futures price and the T-bill yield of $r = 1.0509\%$ as previously to deduce an annualised convenience yield of $\delta = -0.139\%$. The CRR tree for

spot gold was then built using $p = \frac{e^{[(r-\delta)\Delta t]} - d}{u - d}$ and u and d as per usual.

VIII. Results

Table 1 displays the market prices and the model prices for the options for the J-GIBT. The three implied trees (JW-OBJ, RUB-OBJ, SM-OBJ) all price the options to within at least a tenth of a penny. The CRR tree is calibrated using the Black model implied volatility for the at-the-money option, so it does not price the at-the-money option precisely.¹⁶ The CRR model does a poor job fitting the option prices over the full range of option strikes. Each of the models prices the futures to well within a tenth of a penny. Although not shown, ignoring the American-style exercise, the option pricing is not as accurate; early exercise is optimal in some of the upper branches of the futures trees.

(INSERT TABLE 1 HERE)

Table 2 shows the Jackwerth generalized weight function for each method. It is of course linear for the CRR futures tree by definition. Constraining the α to be within lower and upper bounds of 0.7 and 1.3 times the linear weight function respectively produce well behaved results. Estimating the weight function without constraint often leads to bizarre deviations from optimality. The results in Table 2 are not very far from linearity partly by design.

(INSERT TABLE 2 HERE)

Table 3 shows the value of the real option example from earlier in the paper using a traditional CRR tree built for the spot process using 60 days of London gold fix prices, the three implied binomial trees (J-GIBT with each of JW, RUB, and SM objectives), and the CRR futures tree probabilities. Three values for the number of ounces of gold are given: 4,000, 4,500, and 5,000.

¹⁶ Calibrating the CRR futures tree with a volatility that forces the tree to price the at-the-money option precisely does not change any of the other qualitative properties of our results.

(INSERT TABLE 3 HERE)

Before discussing the value of the real option, it is necessary to discuss the relative shapes of the distributions (see Figures 1, 2, and 3).

(INSERT FIGURES 1, 2, and 3)

First, the implied JW-OBJ distribution of the spot gold price at day-100 is not credible (see Figure 2).¹⁷ Second, the implied probability distributions for the RUB-OBJ and SM-OBJ optimizations are similar (see Figures 1 and 2) and each displays more peakedness than the CRR-FUT lognormal (i.e., the CRR futures tree lognormal distribution built using the ATM implied volatility from the Black model). Third, Figure 3 shows that the CRR-FUT distribution is more peaked in turn than is the CRR-SPOTFIX distribution (i.e., the CRR spot tree built using gold fix prices and estimated volatility). Finally, note that the 69-day implied distributions for the RUB-OBJ and SM-OBJ optimizations are similar, however, the JW-OBJ results appear to be “not credible” given the 100-day results.

There is little difference between the real option pricing in Table 3 from the RUB-OBJ and SM-OBJ optimizations. Both, however, give a real option valuation that is higher, then lower, then higher than the CRR-SPOTFIX model at low, medium, and high gold recovery rates respectively (see Table 3). This non-linear behavior is due to the shape of the implied distributions that contains information not captured by the lognormal. The two CRR valuations in Table 3 are quite different from one another. The

¹⁷ All three objectives (RUB, JW, SM) give well behaved probability distributions at option maturity (Figure 2), but the Jackwerth objective alone gives spiky distributions at futures maturity (Figure 1). We think this spiky distribution of future spot prices is because of the interaction of the extrapolation out to the futures maturity and the presence of the nonlinear option prices in the objective. We occasionally arrived at spiky distributions for the other objectives if the probabilities were not properly constrained to be positive, but we got nothing but this with the JW-OBJ no matter how constrained.

CRR estimation using the gold fix prices arrives at a higher estimated volatility of gold than that implied by the Black model calibration (this can be seen both in the volatility numbers previously mentioned and in the shape of the cumulative distribution functions in Figure 3).

Our IBT models were calibrated using only every second gold futures option strike over a range. Table 4 displays the out-of-sample gold futures options pricing from each of the futures pricing models applied to the remaining gold futures options strikes in the same range. The pricing is quite similar between the three IBT objective functions and is generally within one or two percent of the market price (except for the furthest out of the money case).

(INSERT TABLE 4 HERE)

We may *summarise* our results as follows. First, the J-GIBT captures non-lognormal properties of the underlying that are not reflected in either a CRR tree for the future spot price, or in a CRR tree for the underlying based on the gold fix. Second, the Rubinstein and Smooth objective functions are better behaved in the analysis than the Jackwerth objective function. Third, the implied trees can be calibrated to provide very accurate pricing of traded options and the Jackwerth generalized weight function works well and is not far from linear. Finally, real option analysis that reflects a market consensus of the future (implied from traded securities) provides very different answers relative to the “standard” CRR analysis.

IX. Conclusion

Overall, the contributions of this paper to the literature are three-fold. First, it is the first application of an implied binomial tree (IBT) based on options traded on futures and not traded on the spot security. This allows for calibration of option pricing models to non-lognormal market consensus probability distributions of commodity prices. Second, real options analysis for commodities can now be based on IBTs using securities with very long maturities. One of the criticisms of real option analysis is that it requires volatility measures that cannot be deduced for an extended time period. This presents a problem for capital budgeting decisions with long planning horizons. By developing this technique to incorporate commodity futures options, much longer maturity securities are now available for real option analysis. The third benefit is the comparison of the different implied binomial tree models with different objective functions. Such empirical comparisons do not exist in the current literature.

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Table 1: For the six gold futures options with market prices as listed, the CRR tree of futures prices gives the option prices labelled “CRR-FUT,” whereas the J-GIBT yields the prices given for each of the three objective functions used (Jackwerth, Rubinstein, and Smooth, labelled “JW-OBJ,” “RUB-OBJ,” and “SM-OBJ,” respectively.). On May 19, 2004, the futures options have 69 days to maturity, and the underlying futures have 100 days to maturity.

STRIKE	\$ 360	\$ 370	\$ 380	\$ 390	\$ 400	\$ 410	FUT
MKT	\$ 27.500	\$ 19.700	\$ 13.700	\$ 9.400	\$ 6.300	\$ 4.300	\$ 384.000
CRR-FUT	\$ 26.795	\$ 19.414	\$ 13.256	\$ 8.595	\$ 5.197	\$ 2.955	\$ 384.000
JA-OBJ	\$ 27.498	\$ 19.700	\$ 13.704	\$ 9.399	\$ 6.300	\$ 4.297	\$ 384.000
RU-OBJ	\$ 27.500	\$ 19.700	\$ 13.700	\$ 9.400	\$ 6.300	\$ 4.300	\$ 384.000
SM-OBJ	\$ 27.500	\$ 19.700	\$ 13.700	\$ 9.400	\$ 6.300	\$ 4.300	\$ 384.000

Table 2: This table shows the estimated values of the Jackwerth generalized weight function for each of the three objective functions used (Jackwerth, Rubinstein, and Smooth, labeled “JW-OBJ,” “RUB-OBJ,” and “SM-OBJ,” respectively).

	CRR (LINEAR)	RUB-OBJ	JW-OBJ	SM-OBJ
α_0	0 (fixed)	0 (fixed)	0 (fixed)	0 (fixed)
α_1	0.100	0.100	0.103	0.100
α_2	0.200	0.200	0.194	0.200
α_3	0.300	0.322	0.374	0.301
α_4	0.400	0.354	0.346	0.350
α_5	0.500	0.534	0.357	0.536
α_6	0.600	0.519	0.741	0.514
α_7	0.700	0.653	0.874	0.700
α_8	0.800	0.787	0.837	0.800
α_9	0.900	0.890	0.846	0.900
α_{10}	1 (fixed)	1 (fixed)	1 (fixed)	1 (fixed)

Table 3: This table shows the value of the gold mine real option for each of the implied techniques and for a range of gold recovery values. “CRR-SPOTFIX” is the valuation performed using a CRR tree of spot gold prices based on parameters derived from the London gold fix. The rows labelled “JW-OBJ,” “RUB-OBJ,” and “SM-OBJ” are the values of the option using the tree of spot prices and probabilities derived from the J-GIBT optimization for each of the three objective functions used (Jackwerth, Rubinstein, and Smooth, respectively). The row labelled “CRR-FUT” is the real option valuation based on a tree of spot gold prices using the ending nodal probabilities from the CRR tree for the futures.

	4000oz	4500oz	5000oz
CRR-SPOTFIX	\$2,037.60	\$38,772.05	\$166,159.62
JW-OBJ	\$5,296.62	\$28,624.73	\$166,889.61
RUB-OBJ	\$8,583.57	\$36,846.68	\$168,321.82
SM-OBJ	\$9,045.59	\$36,877.92	\$168,418.98
CRR-FUT	\$875.31	\$31,812.01	\$162,299.16

Table 4: Out of Sample Pricing using the CFO pricing models and the settlements of the in-between call option strikes previously omitted.

STRIKE	\$ 365	\$ 375	\$ 385	\$ 395	\$ 405	\$ 415	FUT
MKT	\$ 23.100	\$ 16.500	\$ 11.300	\$ 7.700	\$ 5.200	\$ 3.500	\$ 384.000
CRR-FUT	\$ 22.965	\$ 16.155	\$ 10.773	\$ 6.706	\$ 3.974	\$ 2.162	\$ 384.000
JA-OBJ	\$ 23.430	\$ 16.456	\$ 11.349	\$ 7.850	\$ 5.180	\$ 3.414	\$ 384.000
RU-OBJ	\$ 23.418	\$ 16.436	\$ 11.380	\$ 7.711	\$ 5.189	\$ 3.616	\$ 384.000
SM-OBJ	\$ 23.419	\$ 16.434	\$ 11.380	\$ 7.710	\$ 5.187	\$ 3.624	\$ 384.000

Figures 1A and 1B: This figure shows the implied nodal density functions for the futures prices underlying the futures options along with the CRR futures distribution. These are the distributions at t=100 days (maturity of the futures). They are thus also implied distributions of spot prices. Three different objective functions are used in the J-GIBT optimization. These are labelled “RUB-OBJ,” “JW-OBJ,” and “SM-OBJ,” and refer to the Rubinstein, Jackwerth, and Smooth objective functions, respectively. Figure 1A shows RUB-OBJ and SM-OBJ along with the CRR futures; Figure 1B shows JW-OBJ along with the CRR futures.

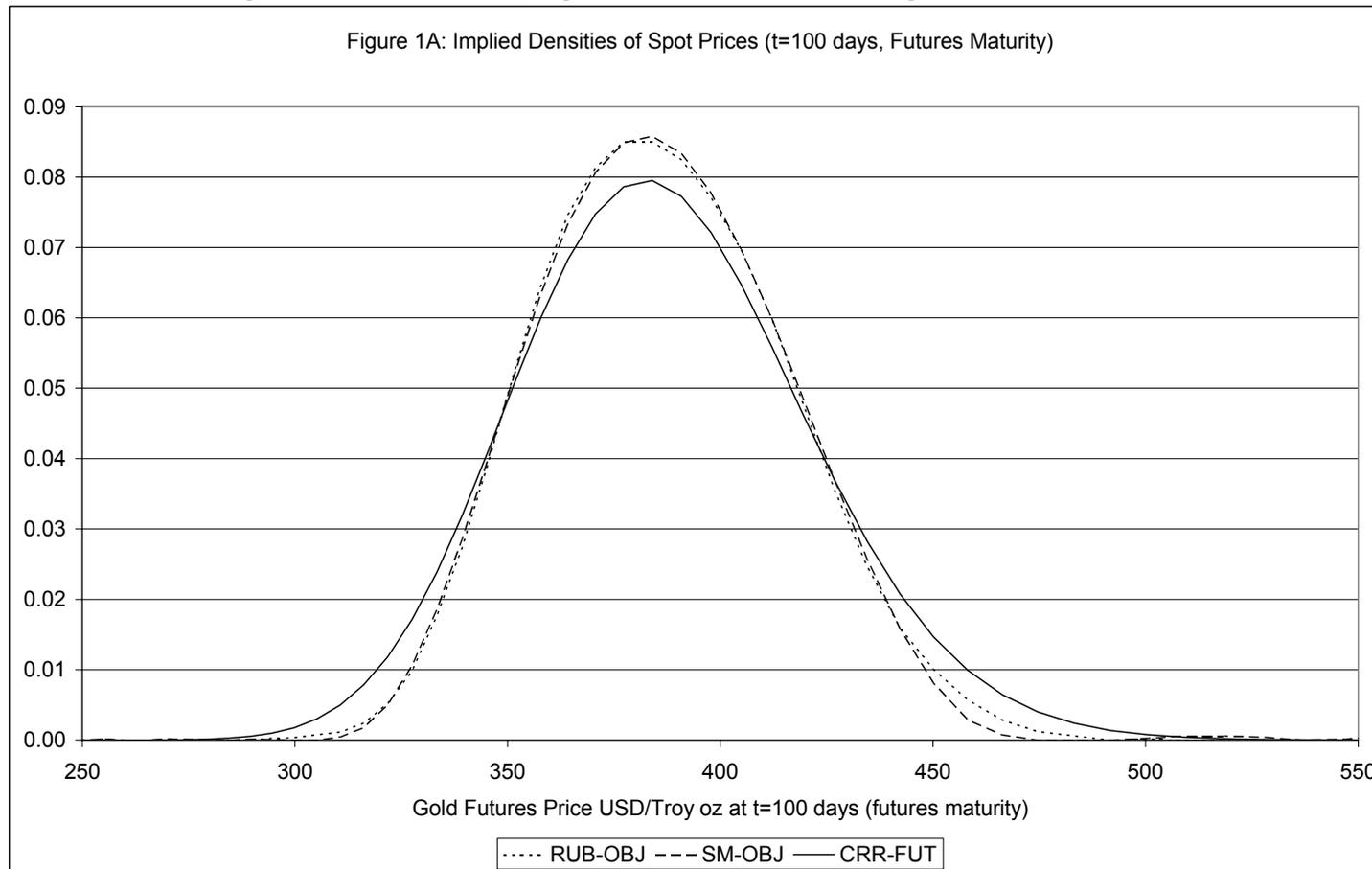


Figure 1B: Implied Densities of Spot Prices (t=100 days, Futures Maturity)

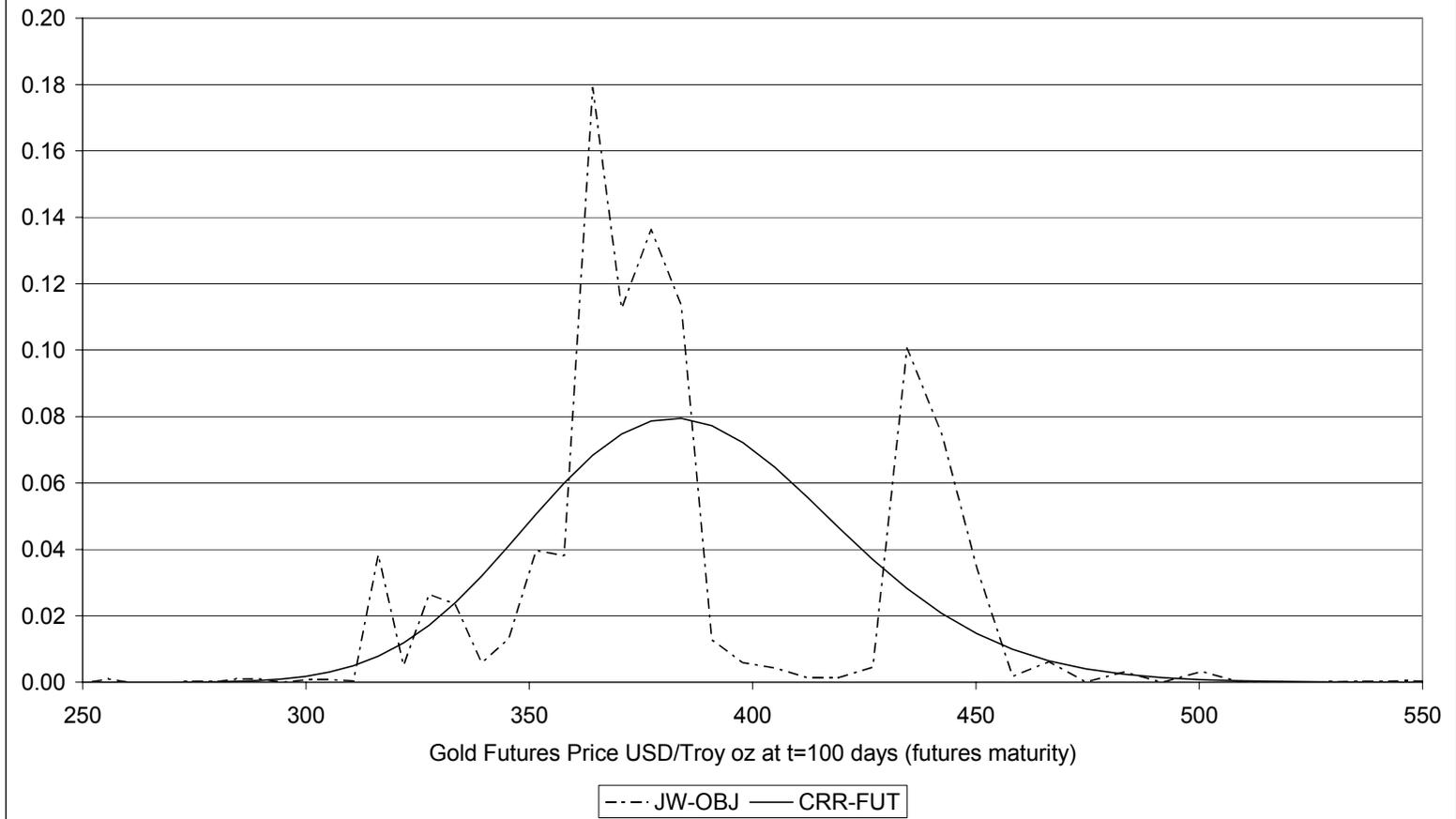


Figure 2: This figure shows the implied nodal density functions for the futures prices underlying the futures options. Although the Jackwerth generalized implied binomial tree (J-GIBT) is applied at $t=100$ days (maturity of the futures), the distributions shown here are those that the $t=100$ optimization leads to at $t=69$ days (maturity of the options). Three different objective functions are used in the J-GIBT optimization. These are labelled “RUB-OBJ,” “JW-OBJ,” and “SM-OBJ,” and refer to the Rubinstein, Jackwerth, and Smooth objective functions, respectively. Although indicative, there is a slight approximation in the figure because each implied technique has slightly different underlying values at $t=69$, and these in turn differ from the CRR (not implied) futures distribution.

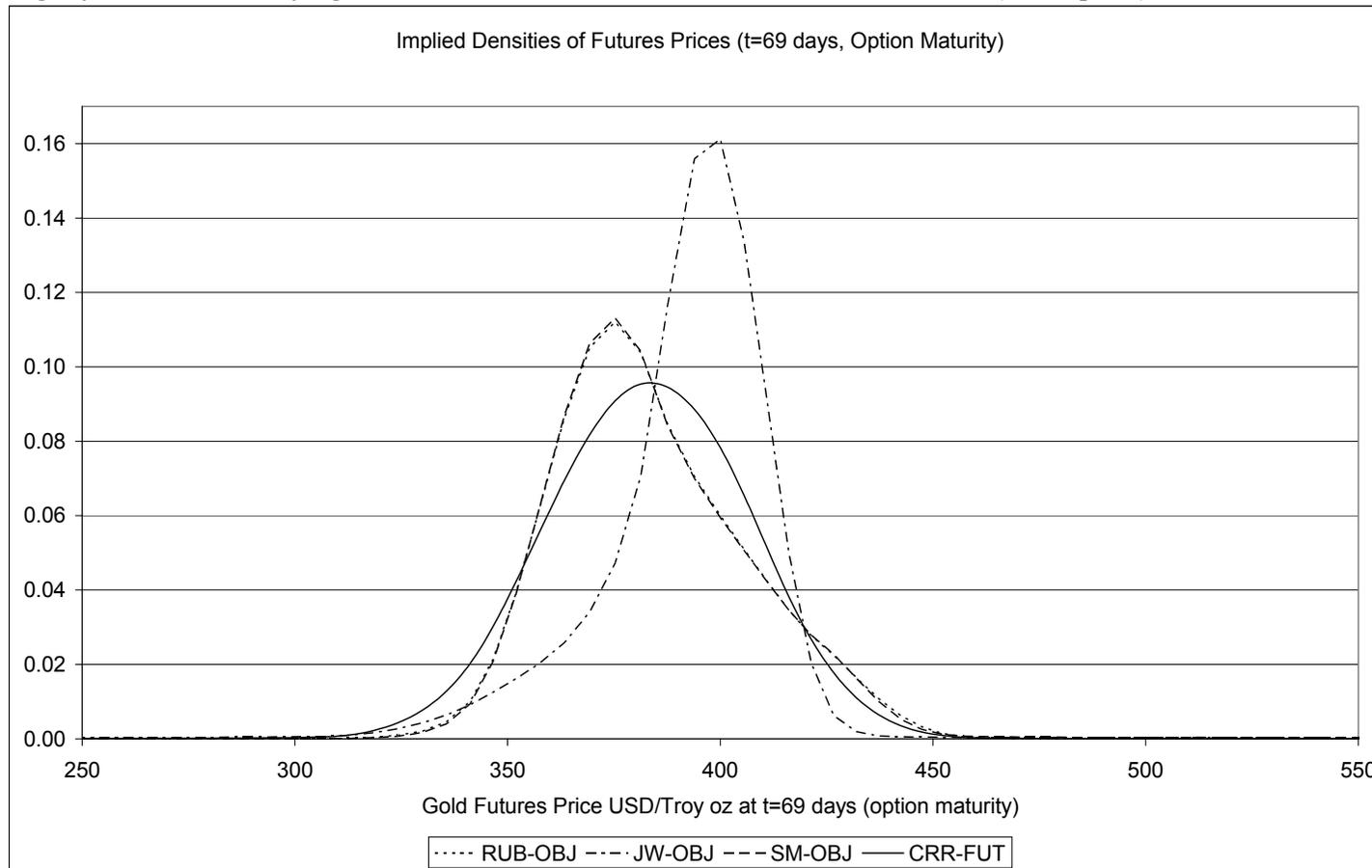


Figure 3: This figure shows the ending nodal (cumulative) density functions (cdf) for the CRR tree of futures prices at $t=100$ days (also equal to spot prices by definition), and for the CRR tree of spot prices built using the gold fixing data. They are labelled “CRR-FUT” and “CRR-SPOTFIX,” respectively. They are shown as cdfs because the discrete nature of the nodal densities does not allow meaningful illustration of the raw data without smoothing.

