ESTIMATING BETAS: FIRM SPECIFIC ESTIMATES, INDUSTRY AVERAGES AND COMBINED ESTIMATES

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1. Introduction

In estimating the beta of an individual firm, the generally employed practice is to average across the estimates for a set of firms within the relevant industry. However, whilst the resulting estimate may be reliable in respect of the average true beta of this set of firms, it is likely to be less reliable in respect of the firm of concern because the true beta of this firm is likely to differ from the industry average. Mindful of this problem, and invoking Bayesian concepts, Vasicek (1973) has proposed an estimate that is a weighted average of an OLS estimate for the firm of concern and the mean of a prior distribution, with weights governed by the variance of the OLS estimate for the firm of concern and the variance of the prior distribution. Typical applications of this model equate the prior distribution with a set of beta estimates from the relevant industry, in which case the variance of the prior distribution is estimated by the cross-sectional variance in these estimates. Lally (1998a) critiques this approach on the grounds that the prior distribution is the distribution of *true* betas rather than their estimates, and the cross-sectional variance in these true betas will be appreciably less than the cross-sectional variation in the estimated betas. Furthermore, Vasicek's estimator assumes that the data from which the prior distribution is inferred is statistically independent of the OLS estimate for the individual firm of concern. This would be achieved if the data from which the prior distribution is inferred predated the data underlying the OLS estimate for the firm of concern, but this would require data of the first type that was some years old[1]. Given that betas change over time, and therefore the use of current data is preferred, a superior approach would be to combine an OLS estimate for the firm of concern with a current estimate for the industry average, and this will require recognition of correlation between the two underlying estimators.

In light of all this, the present paper has two goals. The first is to develop a Vasicek-like estimator for the beta of a firm, involving a current OLS estimator for the firm of concern and a current estimator for the industry average; we call this the combined estimator. The second goal is to empirically assess the advantage (in terms of reduced variance in the estimator) from using the combined estimator over that of its two

[1] For example, if the OLS estimate for the firm of concern is based upon the last five years data, then the data from which the prior distribution is inferred would have to be at least five years old.

components. If the advantage over the industry average is slight, and the industry average is superior to the OLS estimate for the firm of concern, then the usual practice of invoking the industry average would be vindicated.

2. Theory

Consistent with standard practice in using data from other firms, we estimate the asset beta of the firm of concern; consequently, all of the following analysis is framed in terms of asset betas[2]. Define β as the average true asset beta for a set of firms in the same industry, $\bar{\hat{\beta}}$ as the average estimated asset beta of this set of firms and β_1 as the true asset beta for one of these firms (the firm of concern). Assuming that we have no information about the divergence of β_j from β, then β_j can be expressed as $\beta_j = \beta + d_j$, where d_j is a random drawing from the cross-sectional distribution of the firms' true asset betas (with variance denoted σ_d^2). Letting e_j denote the beta estimation error for firm j, it follows that the estimated asset beta for firm j is as follows.

$$\hat{\beta}_j = \beta_j + e_j = \beta + d_j + e_j$$

Defining k as the weighting applied to $\hat{\beta}_1$, the combined estimator is then as follows

$$k\hat{\beta}_1 + (1-k)\bar{\hat{\beta}}$$

and the error associated with this estimator for β_1 is then as follows.

$$k\hat{\beta}_1 + (1-k)\bar{\hat{\beta}} - \beta_1$$

[2] Having estimated the asset beta for the firm of concern, an adjustment for leverage must then be undertaken, such as that in Hamada (1972), so as to generate an estimate for the equity beta of the firm of concern. However, the present paper has nothing to say about the second step, and it is therefore omitted. By contrast, the analysis in Vasicek (1973) focuses directly upon the equity beta rather than upon the asset beta followed by the gearing correction. Lally (1998a) shows that the two-step process is superior, because it avoids attributing cross-sectional variation in leverage (and therefore some of the cross-sectional variation in estimated equity betas) to estimation error.

For a randomly selected firm 1, this error is mean zero. So, in choosing the weighting k, an appropriate goal would be to minimise the variance of this error. Let N denote the number of firms used in forming the industry average. Recognising that the random variables $d_1 \ldots. d_N$ are mutually independent, that d_j and e_j are independent for each j, and that the random variables $e_1 \ldots. e_N$ are correlated with correlation coefficient denoted ρ, this variance is as follows[3].

$$Var[k\hat{\beta}_1 + (1-k)\bar{\hat{\beta}} - \beta_1] = Var[k(\beta + d_1 + e_1) + (1-k)\frac{1}{N}\sum_{j=1}^{N}(\beta + d_j + e_j) - (\beta + d_1)]$$

$$= Var\left[d_1\left(k + \frac{1-k}{N} - 1\right) + (1-k)\frac{1}{N}\sum_{j=2}^{N} d_j + e_1\left(k + \frac{1-k}{N}\right) + (1-k)\frac{1}{N}\sum_{j=2}^{N} e_j\right]$$

$$= \sigma_d^2\left[\left(k + \frac{1-k}{N} - 1\right)^2 + \frac{(1-k)^2}{N^2}(N-1)\right] + \sigma_e^2\left[\left(k + \frac{1-k}{N}\right)^2 + \frac{(1-k)^2}{N^2}(N-1)\right]$$

$$+ 2\rho\sigma_e^2\left[\left(k + \frac{1-k}{N}\right)\left(\frac{1-k}{N}\right)(N-1)\right] + \rho\sigma_e^2\frac{(1-k)^2}{N^2}\left[(N-1)^2 - (N-1)\right]$$

$$= \sigma_d^2(1-k)^2\left(1 - \frac{1}{N}\right) + \sigma_e^2\left[\frac{1}{N} + k^2\left(1 - \frac{1}{N}\right)\right] + \rho\sigma_e^2\left(1 - k^2\right)\left(1 - \frac{1}{N}\right) \qquad (1)$$

Differentiating this function with respect to k, setting it to zero and solving for k then yields the following.

$$k = \frac{\sigma_d^2}{\sigma_d^2 + \sigma_e^2(1-\rho)} \qquad (2)$$

This result is remarkably close to that in Vasicek (1973), differing only in the presence of the correlation coefficient ρ. Implementation of this model requires estimates for σ_d^2, σ_e^2 and ρ. The first of these cannot be directly estimated, but can be deduced from the cross-sectional distribution of the estimated betas for the N firms. Defining V as the expectation of the cross-sectional sample variance in the estimated betas, it follows that

[3] The analysis down to equation (4) is an extension of that in Boyle et al (2006), which is concerned with the variance of the estimator for the industry average beta.

$$V = E\left[\frac{\sum_{j=1}^{N}(\hat{\beta}_j - \overline{\hat{\beta}})^2}{N-1}\right]$$

$$= \frac{1}{N-1}\sum_{j=1}^{N} E(\hat{\beta}_j - \overline{\hat{\beta}})^2$$

$$= \frac{N}{N-1} Var(\hat{\beta}_j - \overline{\hat{\beta}})$$

Without loss of generality, let firm j be firm 1. It follows that

$$V = \frac{N}{N-1} Var\left[\hat{\beta}_1 - \frac{1}{N}\sum_{j=1}^{N}\hat{\beta}_j\right]$$

$$= \frac{N}{N-1} Var\left[\beta + d_1 + e_1 - \frac{1}{N}\sum_{j=1}^{N}(\beta + d_j + e_j)\right]$$

$$= \frac{N}{N-1} Var\left[d_1\left(\frac{N-1}{N}\right) + e_1\left(\frac{N-1}{N}\right) - \frac{1}{N}\sum_{j=2}^{N} d_j - \frac{1}{N}\sum_{j=2}^{N} e_j\right]$$

Recognising that the random variables d_j and e_j are independent, and that $d_1 \ldots\ldots d_N$ are mutually independent, it follows that

$$V = \frac{N}{N-1}\left[\sigma_d^2\left(\frac{N-1}{N}\right)^2 + \frac{\sigma_d^2}{N^2}(N-1)\right] + \frac{N}{N-1} Var\left[e_1\left(\frac{N-1}{N}\right) - \frac{1}{N}\sum_{j=2}^{N} e_j\right]$$

$$= \sigma_d^2 + \frac{N}{N-1}\left[\sigma_e^2\left(\frac{N-1}{N}\right)^2 + \frac{\sigma_e^2}{N^2}(N-1)\right]$$

$$+ \frac{N}{N-1}\rho\sigma_e^2\left[-2\left(\frac{N-1}{N}\right)\frac{1}{N}(N-1) + \frac{(N-1)^2 - (N-1)}{N^2}\right]$$

$$= \sigma_d^2 + \sigma_e^2 - \rho\sigma_e^2 \qquad (3)$$

Solving this equation for σ_d^2 yields the following result.

$$\sigma_d^2 = V - \sigma_e^2(1-\rho) \qquad (4)$$

Estimates for V and also for σ_e^2 arise from the set of estimates for the asset betas of the N firms, and these in turn derive from the OLS estimates for the equity betas. However, each regression generates an estimate of an equity beta, and corrections for leverage are required to produce an estimate of an asset beta. Letting $\hat{\beta}_{ej}$ denote the estimated equity beta for firm j arising from the regression, and following Hamada (1972), the estimate for the asset beta of the firm is as follows

$$\hat{\beta}_j = \frac{\hat{\beta}_{ej}}{\left[1+\frac{L}{1-L}(1-T_c)\right]} \tag{5}$$

where L is market leverage and T_c is the corporate tax rate. It then follows that the variance of the estimation error for the asset beta of firm j (which is assumed to be equal across all firms in the set examined) is as follows.

$$\sigma_e^2 = Var(\hat{\beta}_j) = \frac{Var(\hat{\beta}_{ej})}{\left[1+\frac{L}{1-L}(1-T_c)\right]^2} \tag{6}$$

In respect of the correlation coefficient ρ, the regression results will not provide an estimate of this. However, an estimate can still be obtained by consideration of the regression model that underlies the beta estimation process. Let R_{jt} denoting firm j's unlevered return in period t, R_{mt} the market return in period t, and u_{jt} the firm specific return component for that period:[4]

$$R_{jt} = a_j + \beta_j R_{mt} + u_{jt} \tag{7}$$

The OLS estimate of β_j is then as follows.

[4] We assume here that the regression process directly estimates the asset beta. However, as noted above, the regression process directly estimates the equity beta, from which the estimate of the asset beta arises by an adjustment for leverage. This detail is not significant to the issue examined here.

$$\hat{\beta}_j = \frac{\hat{C}ov(R_j, R_m)}{\hat{V}ar(R_m)} = \frac{\beta_j \hat{C}ov(R_m, R_m) + \hat{C}ov(u_j, R_m)}{\hat{V}ar(R_m)} = \beta_j + \frac{\hat{C}ov(u_j, R_m)}{\hat{V}ar(R_m)}$$

So the error in estimating the asset beta for firm j is as follows.

$$\begin{aligned} e_j &= \hat{\beta}_j - \beta_j \\ &= \frac{\hat{C}ov(u_j, R_m)}{\hat{V}ar(R_m)} \\ &= \frac{\sum_{t=1}^{T}(u_{jt} - \overline{u}_j)(R_{mt} - \overline{R}_m)}{\hat{V}ar(R_m)} \\ &= \frac{\sum_{t=1}^{T}\tilde{u}_{jt}\tilde{R}_{mt}}{\hat{V}ar(R_m)} \end{aligned} \tag{8}$$

where the tilde indicates that the variable in question is expressed as the difference from its sample mean. Consequently, the estimation errors e_j will be independent across firms so long as the regression residuals u_{jt} are independent across firms. However, "industry effects" imply that the regression residuals will be positively correlated amongst firms in the same industry, and therefore the estimation errors e_j will have the same property. Using equation (8), it follows that

$$\begin{aligned} \rho &= Corr(e_i, e_j) \\ &= \frac{Cov(e_i, e_j)}{\sqrt{Var(e_i)Var(e_j)}} \\ &= \frac{Cov(e_i, e_j)}{Var(e_j)} \end{aligned}$$

$$= \frac{Cov\left(\sum_{t=1}^{T} \tilde{u}_{it}\tilde{R}_{mt}, \sum_{t=1}^{T} \tilde{u}_{jt}\tilde{R}_{mt}\right)}{Var\left(\sum_{t=1}^{T} \tilde{u}_{jt}\tilde{R}_{mt}\right)}$$

Recognising that the random variables $\tilde{u}_{it}$ and $\tilde{R}_{mt}$ are each serially independent, it follows that

$$\rho = \frac{Cov\left(\tilde{u}_i\tilde{R}_m, \tilde{u}_j\tilde{R}_m\right)T}{Var\left(\tilde{u}_j\tilde{R}_m\right)T}$$

Recognising that each of the random variables here is mean zero, and that both $\tilde{u}_i$ and $\tilde{u}_j$ are independent of $\tilde{R}_m$, it follows that

$$\rho = \frac{E\left(\tilde{u}_i\tilde{u}_j\tilde{R}_m^2\right)}{E\left(\tilde{u}_j^2\tilde{R}_m^2\right)}$$

$$= \frac{E\left(\tilde{u}_i\tilde{u}_j\right)E\left(\tilde{R}_m^2\right)}{E\left(\tilde{u}_j^2\right)E\left(\tilde{R}_m^2\right)}$$

$$= \frac{Cov(\tilde{u}_i, \tilde{u}_j)}{Var(\tilde{u}_j)}$$

$$= \frac{Cov(u_i, u_j)}{Var(u_j)}$$

If we represent u_i as the sum of an industry effect (I) and an uncorrelated firm-specific effect θ_j, it then follows that

$$\rho = \frac{Cov(I + \theta_i, I + \theta_j)}{Var(u_j)}$$

$$= \frac{Var(I)}{Var(u_j)} \tag{9}$$

So, the correlation coefficient ρ is the proportion of $Var(u_j)$ that is explained by the "industry effect". Implicit in this development is the assumption that the industry effect is the same for all firms. If, as seems likely, this is not the case, then ρ will differ across pairs of firms. However, it does not seem to be possible to estimate values for ρ that vary across firm pairs. Consequently, it is necessary to assume that the industry effect is the same across all firms within the relevant industry.

3. Comparison of Estimators

The previous section has determined the variance of the combined estimator, the optimal value for k and procedures for estimating other relevant parameters. This section now seeks to estimate this variance and to compare it with the variance arising from use of the industry average ($k = 0$) and use of the OLS estimate for the firm of concern ($k = 1$). To empirically assess the issues arising from this theoretical analysis, we consider two industries: US electric utilities and gas distribution firms. In particular, we draw upon estimated equity betas and their standard errors from S&P, for the periods 1994-1998 and 1999-2003. These estimated equity betas and their standard errors are converted to estimated asset betas and standard errors using equations (5) and (6). Following standard practice, L is defined as that prevailing at the end of the estimation period. The corporate tax rate used is .39, comprising the Federal rate of .34 and an allowance of .04 for state taxes (Tax Foundation, 2005).

For each of these two industries and five-year spans, an unbiased estimate for V will arise from the cross-sectional sample variance in the estimated asset betas. These four estimates for V are shown in the fourth column of Table 1 below.[5] Turning now to σ_e^2, an unbiased estimate arises from each of the regressions that generates an estimate of the asset beta. For each of the four sets of firms, we average over the estimates for the individual firms (with the same deletions noted in footnote 5), and the results are shown in Table 1 below.

In respect of US utilities, King (1966, Table 9) shows that around 30% of the variance in the regression residual u_j is explained by industry effects. This implies an estimate for ρ of .30. Using a larger data set, Meyers (1973, Table 1) generates the lower estimate of .12, but this is for the market as a whole. Nevertheless, it can be

[5] For the second of these sets (electric utilities for 1994-1998), data from one of the 37 firms was disregarded on the grounds of being an exceptional outlier. For each of the last two sets, one firm was deleted for the same reason.

compared with King's market-wide estimate of .26, and suggests an estimate for ρ of .14 for US utilities. Giving somewhat more weight to the more recent of the two studies suggests an estimate for ρ of about .20.

Table 1: The Variances of the Estimates

		N	V	$\hat{\sigma}_e^2$	$\hat{\sigma}_d^2$	k_0	*Var*(*estimate*) $k = 1$	$k = k_0$	$k = 0$
Electric	1994-1998	37	.0135	.0147	.0017	.13	.0147	.0047	.0049
Electric	1999-2003	42	.0734	.0273	.0516	.70	.0273	.0210	.0564
Gas	1994-1998	36	.0450	.0185	.0302	.67	.0185	.0138	.0335
Gas	1999-2003	38	.0368	.0284	.0141	.38	.0284	.0112	.0200
Mean			.0422	.0222	.0244	.47	.0222	.0127	.0287

We are now in a position to estimate the variance of the combined estimator. Using the estimates for V and σ_e^2 shown in the fourth and fifth columns of Table 1 above, along with the estimate for ρ of .20, the resulting estimates of σ_d^2 in accordance with equation (4) are shown in the sixth column of Table 1. Interestingly, these estimates are approximately as large as the variances for the OLS estimates for individual firms, i.e., there is approximately as much cross-sectional variation in the true asset betas of firms within an industry as there is in estimating the asset beta of an individual firm. The optimal value of k is then determined in accordance with equation (2), and shown in the next column of the table (k_0). Finally, the variance of the combined estimator is determined in accordance with equation (1) and the results are shown in the penultimate column of Table 1. By way of comparison, estimates of the variance arising from values for k of 1 and 0 are also shown in the adjoining two columns. Naturally, the variance using the optimal value for k is always the lowest. In fact, averaged over the four data sets, the variance of this combined estimator is over 40% less than that for an individual firm and over 50% lower than that for the industry average. Remarkably, the use of data for only the firm of concern is generally superior to use of the industry average, and part of the reason for this lies in the substantial variation in true asset betas across firms within the same industry.

The process described above invokes equations (5) and (6) and, following standard practice, defines leverage L to be that prevailing at the end of the five year period used to estimate the equity betas. However Lally (1998b) shows that a superior estimate of the asset beta arises by using the average leverage over the beta estimation period to convert the estimated equity beta into an estimated asset beta, because the OLS estimate of the equity beta will reflect the average leverage rather than the terminal leverage. Adopting this alternative approach may not exert much effect upon the average estimated asset beta but it is liable to materially reduce the cross sectional variance of the estimated asset betas, and the intuition is as follows. Suppose a firm has leverage of L_0 over most of the beta estimation period, but leverage significantly rises (or falls) shortly before the end of this period. Its estimated equity beta will not be materially affected by this event, and its true asset beta will be invariant to it. However, if the estimated equity beta is converted into an estimate of the asset beta using terminal leverage, the estimated asset beta following equation (5) will be significantly lower (or higher) than otherwise. Across a large number of firms, the average estimated asset beta may not be materially affected but the cross-sectional variance in these estimated asset betas will be greater than otherwise. By contrast, in using average leverage to implement equation (5), these spurious effects will be mitigated.

In light of this issue, we now estimate $\hat{\beta}_j$ and $Var(\hat{\beta}_j)$ from equations (5) and (6) using average rather than terminal leverage. The analysis in Table 1 is then repeated and the results shown in Table 2. As expected, the estimates for V are in general significantly less, with the average value declining from .0422 to .0292. Consequently, the estimates for σ_d^2 have the same property, with the average value declining from .0244 to .0128. The variance associated with the industry average (k = 0) is then generally less than before, with the average declining from .0287 to .0170. In turn, this induces a reduction in the variance of the combined estimator, although the extent of this is much more limited. The reduction in the variance of the industry average also leads to this estimator now dominating the OLS estimate for the firm of concern. Of course, both of these are still inferior to the estimate based upon the optimal value for *k*, with the variance of the latter on average less than even the better of the two alternative estimators by over 30%.

Table 2: The Variances of the Estimates

		N	V	$\hat{\sigma}_e^2$	$\hat{\sigma}_d^2$	k_0	$Var(estimate)$ $k = 1$	$k = k_0$	$k = 0$
Electric	1994-1998	37	.0110	.0122	.0012	.11	.0122	.0037	.0039
Electric	1999-2003	42	.0355	.0235	.0167	.47	.0235	.0138	.0215
Gas	1994-1998	36	.0273	.0186	.0124	.45	.0186	.0107	.0161
Gas	1999-2003	38	.0429	.0274	.0210	.49	.0274	.0165	.0265
Mean			.0292	.0204	.0128	.38	.0204	.0112	.0170

4. Conclusions

This paper has developed a Vasicek-like estimator for a firm's asset beta, which involves optimally combining an OLS estimate for the firm of concern and an estimator for the industry average beta. Unsurprisingly, the estimator and its variance are very similar to that of the Vasicek estimator, differing only in recognising correlation between the beta estimates for individual firms.

This paper has also compared the combined estimator to that of its two components, and the conclusions are as follows. Firstly, the combined estimator places about 40% weight upon the OLS estimator for the firm of concern. Secondly, the combined estimator has a variance that is over 30% less than that of the better of the two component estimators; consequently, the usual practice of invoking the industry average is significantly inferior to the use of this combined estimator. Thirdly, in converting an OLS estimate of an equity beta into an estimate for the firm's asset beta, the use of average leverage over the beta estimation period rather than the usual practice of invoking leverage at the end of the estimation period markedly reduces the variance in the estimator for the industry average beta, and this benefit flows through to the combined estimator.

REFERENCES

Boyle. G., Evans, L. and Guthrie, G. 2006, 'Estimating the WACC in a Regulatory Setting', *Institute for the Study of Competition and Regulation* (**www.iscr.org.nz**).

Hamada, R. 1972, 'The Effect of the Firm's Capital Structure on the Systematic Risk of Common Stocks', *The Journal of Finance*, vol. 27, pp. 435-52.

King, B. 1966, 'Market and Industry Factors in Stock Price Behaviour', *Journal of Business*, vol. 39, pp. 139-170.

Lally, M. 1998a, 'An Examination of Blume and Vasicek Betas', *The Financial Review*, vol. 33, pp. 183-197.

________ 1998b, 'Correcting Betas for Changes in Firm and Market Leverage', *Pacific Accounting Review*, vol. 10, pp. 98-115.

Meyers, S. 1973, 'A Re-Examination of Market and Industry Factors in Stock Price Behaviour', *Journal of Finance*, vol. 28, pp. 695-705.

Tax Foundation. 2005, *The US Corporate Tax System: Once a World Leader, Now a Millstone Around the Neck of American Business*, November, Report No. 136.

Vasicek, O. 1973, 'A Note on Using Cross-Sectional Information in Bayesian Estimation of Security Betas', *Journal of Finance*, vol. 28, pp. 1233-1239.