An Analytical Model for the Break-Even Credit Default Swap Spread with No Counterparty Default Risk in Hull (2009): A Pedagogical Approach

Professor Hull in his textbook [Hull, J. C. Options, Futures, & Other Derivatives, Edition 7, Pearson, 2009] illustrates the determination of the credit default swap (CDS) spread by a numerical example. This paper follows the assumptions used by Professor Hull and provides a reduced form formula for the CDS spread. The CDS spread can be simplified by the Taylor series into a function of two variables, the default probability and the recovery rate. Rearranging this simplified formula, the default probability, expressed as the ratio of the CDS spread and the loss given default, is equivalent to the average default intensity, or, the hazard rate defined in Hull [2009, p.500].

A company's credit default swap spread is the cost per annum for protection against a default by the reference entity. Let s denote the break-even CDS spread. Professor Hull [2009] makes the assumptions that (1) the default probability, λ, during a year conditional on no earlier default is known, (2) defaults always happen halfway through a year, (3) the regular payments, s, on the CDS are made once a year in arrears, (4) the accrual payments, $0.5s$, are made in the event of a default, and (5) the risk-free interest rate, r, and the recovery rate, R, are known and constant throughout the life, N years, of the CDS, (6) there are no counterparty default risks, and (7) the variables λ, r, and R are mutually independent (cf. Hull and White [2000]). Furthermore, the default probability λ is risk-neutral. The risk-free interest rate is not changing during the life of the CDS. Therefore, the risk-free interest rate is also risk-neutral. Then s can be determined when the sum of the present value of the expected payoffs is equal to the sum of the present value of expected regular plus accrual payments, summing from year 1 to year N. Hull [2009] shows that when $N=5$, $\lambda=0.02$, $r=5\%$, and $R=40\%$, the break-even CDS spread s is equal to 0.0124, or, 124 basis points per year.

In this paper, given the above assumptions, we formulate the break-even CDS spread in discrete time according to the procedure outlined in Hull [2009]. To give more flexibility to the occurrence of a credit event, we assume that a credit event can occur at time τ through a year, $0 \leq \tau \leq 1$. For instance, $\tau=0.5$ means the credit event occurs half way through a year, and $\tau=0.75$ means a credit event occurs at the end of the third quarter. We find that the break-even CDS spread, s, is independent of the life of the CDS, N, and can be expressed as a reduced form analytical formula in terms of λ, r, τ, and R. When the formula for the CDS spread is approximated by the Taylor series, we show that the CDS spread is determined mainly by the default probability and the recovery rate. The default probability, expressed as the CDS spread divided by the loss given default, is equivalent to the hazard rate defined in Hull [2009, p.500].

DERIVATION OF THE BREAK-EVEN CDS SPREAD

Suppose that conditional on no earlier default a reference entity has a risk-neutral probability of default λ during a year. In year 1, the default probability is λ and therefore the survival probability is $1-\lambda$. Starting from year $t = 2$, the unconditional default probability $q_t = \lambda(1-\lambda)^{t-1}$ and the unconditional survival probability $p_t = (1-\lambda)^t$, $1 \le t \le N$.

Assume that regular CDS payments are made annually in arrears, the expected recovery rate is R and the risk-free interest rate is r per annum with continuous compounding. Assume further that defaults always happen at time τ ($0 < \tau < 1$) through a year. For example, $\tau = 0.75$ means a default happens at the end of the third quarter. Finally, assume that the life of the CDS is N years and the CDS is settled by cash. This means that, when there is a credit event, the protection buyer receives $1-R$ per dollar of the notional principal.

Let s denote the breakeven CDS rate per dollar of the notional principal. Then s can be determined when the sum of the present value of the expected payoffs is equal to the sum of the present value of expected regular plus accrual payments, summed from year 1 to year N.

Expected Payoff

In each year t, at time $t-1+\tau$, if the reference entity defaults (with probability q_t) the protection buyer receives $1-R$. If the reference entity does not default (with probability $1-q_t$) the protection buyer receives nothing. The present value of the expected payoff in year t is

$$\left[q_t \cdot (1-R) + (1-q_t)\cdot 0\right] e^{-r(t-1+\tau)} = q_t \cdot (1-R)\, e^{-r(t-1+\tau)} = \lambda(1-\lambda)^{t-1} \cdot (1-R)\, e^{-r(t-1+\tau)} \quad (1)$$

Expected Regular Payment

If the reference entity does not default (with survival probability p_t) the protection buyer pays s. If the reference entity defaults (with probability $1-p_t$) the protection buyer pays nothing. The present value of the expected regular payment in year t is

$$\left[p_t \cdot s + (1-p_t)\cdot 0\right] e^{-rt} = \left(p_t \cdot s\right) e^{-rt} = (1-\lambda)^t\, s\, e^{-rt} \quad (2)$$

Expected Accrual Payment

If the reference entity defaults (with probability q_t) at time $t-1+\tau$ the accrued payment is τs in year t. If the reference entity does not default (with probability $1-q_t$) the protection buyer pays nothing. The present value of the expected accrual payment in year t is

$$\left[q_t \cdot (\tau\, s) + (1-q_t)\cdot 0\right] e^{-r(t-1+\tau)} = q_t \cdot (\tau\, s)\, e^{-r(t-1+\tau)} = \left[\lambda\,(1-\lambda)^{t-1}\tau\, e^{-r(t-1+\tau)}\right] s \quad (3)$$

The Break-Even CDS Spread

The break-even CDS spread *s* is determined when the total present value of expected payoffs is equal to the total present value of the expected regular payments plus the present value of the expected accrual payments, summed from year 1 to year *N*, i.e.,

$$\sum_{t=1}^{N} \lambda(1-\lambda)^{t-1}(1-R)\, e^{-r(t-1+\tau)} = \sum_{t=1}^{N} \left\{\left[(1-\lambda)^{t}\cdot s\right]e^{-rt} + \left[\lambda(1-\lambda)^{t-1}\cdot(\tau s)\right]e^{-r(t-1+\tau)}\right\} \qquad (4)$$

Taking out the constants independent of time *t* from both sides of equation (4), we have

$$\left[1-\lambda+\lambda\tau\, e^{r(1-\tau)}\right] s \sum_{t=1}^{N}(1-\lambda)^{t-1}e^{-rt} = \left[\lambda\,(1-R)\, e^{r(1-\tau)}\right] \sum_{t=1}^{N} (1-\lambda)^{t-1}\, e^{-rt} \qquad (5)$$

The factor $\sum_{t=1}^{N}(1-\lambda)^{t-1}e^{-rt}$ is positive and can be cancelled from both sides of equation (5). As a result, the break-even credit default spread, *s*, is independent of the life of the CDS, *N*, and is a function of λ, τ, *r*, and *R*.

$$s = \frac{\lambda\,(1-R)\, e^{r(1-\tau)}}{1-\lambda+\lambda\tau\, e^{r(1-\tau)}} \qquad (6)$$

Let λ=0.02, τ=0.5, *r*=0.05, and *R*=0.4. The CDS spread *s* is equal to 0.012425. The credit default swap is binary when *R* = 0.

SENSITIVITIES OF THE CDS SPREAD

The CDS spread is determined by four endogenous variables. The sensitivities of the CDS spread with respect to the default probability and interest rate are positive while the sensitivities of the CDS spread with respect to the time a credit event occurs through a year and the recovery rate are negative.

$$\frac{\partial s}{\partial \lambda} = \frac{(1-R)\, e^{r(1-\tau)}}{\left(1-\lambda+\lambda\tau\, e^{r(1-\tau)}\right)^2} > 0$$

$$\frac{\partial s}{\partial r} = \frac{\lambda(1-\lambda)(1-\tau)(1-R)\, e^{r(1-\tau)}}{\left(1-\lambda+\lambda\tau\, e^{r(1-\tau)}\right)^2} = \lambda(1-\lambda)(1-\tau)\,\frac{\partial s}{\partial \lambda} > 0$$

$$\frac{\partial s}{\partial \tau} = -\frac{r\lambda(1-\lambda)(1-R)\, e^{r(1-\tau)}}{\left(1-\lambda+\lambda\tau\, e^{r(1-\tau)}\right)^2} = -r\lambda(1-\lambda)\,\frac{\partial s}{\partial \lambda} < 0$$

$$\frac{\partial s}{\partial R} = -\frac{\lambda\, e^{r(1-\tau)}}{1-\lambda+\lambda\tau\, e^{r(1-\tau)}} < 0$$

Since the values of the variables λ, r, τ, and R are all less than unity, the values of $\frac{\partial s}{\partial r}$ and $\frac{\partial s}{\partial \tau}$ are both very small. We use the Taylor series to simplify the expression of the CDS spread and its sensitivities below and find some very interesting properties of the CDS spread.

THE DEFAULT PROBABILITY AND THE HAZARD RATE

Using the Taylor series expansion, the exponential function e^x can be written as $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ The variables λ, τ, and r are positive numbers smaller than unity. Assume that the values of the square and high orders of these variables are negligible. Furthermore, assume that the value of the product among these 3 variables and their higher orders are also negligible. Then

$$s = \frac{\lambda (1-R)\, e^{r(1-\tau)}}{1-\lambda+\lambda\tau\, e^{r(1-\tau)}} \cong \frac{(1-R)\lambda}{1-\lambda} \tag{7}$$

When λ= 0.02 and R=0.4, s=0.012245 which is very close to the exact value 0.0124 using equation (6) with an error of less than 1.5%. From equation (7), we can see that the CDS spread is now determined only by λ and R. This is consistent to the finding in the above section that the values of both $\frac{\partial s}{\partial r}$ and $\frac{\partial s}{\partial \tau}$ are very small.

From equation (7), we note that the CDS spread is proportional to 1–R. Solving for the default probability λ in equation (7), we have $\lambda = \frac{s}{1-R+s}$. The value of the CDS spread s is very small compared with the recovery rate R. The default probability λ can be approximated by

$$\lambda = \frac{s}{1-R} \tag{8}$$

Thus the default probability is inversely proportional to 1–R. Suppose that s is the extra yield offered by a corporate bond over and above a risk-free bond. When s is divided by 1–R, the loss given default, λ is defined as the hazard rate in Hull [2009, p.500]. As a result, the default probability defined in this paper can be estimated by the hazard rate. In equation (8), λ is always larger than s. The ratio of s over λ is equal to 1–R, or, the loss given default.

The sensitivities of the CDS spread with respect to the default probability and the recovery rate are now:

$$\frac{\partial s}{\partial \lambda} = \frac{1-R}{(1-\lambda)^2} > 0 \text{ and } \frac{\partial s}{\partial R} = -\frac{\lambda}{1-\lambda} < 0.$$

CONCLUSION

In this paper, we derive a reduced form formula for the CDS spread in terms of the default probability, interest rate, recovery rate, and the time a credit event occurring through a year. The CDS spread is independent of the life of the CDS. Using the Taylor series expansion, we find that the CDS spread is determined mainly by the default probability and the recovery rate. The CDS spread is directly related to $1 - R$ while the default probability is inversely related to $1 - R$. The CDS spread derived in this paper is consistent with the hazard rate. For a binary CDS, the CDS spread can be approximated by the default probability.

The formula derived in this paper applies to a highly simplified financial environment in which both the riskless interest rate and the default intensity are constant over time. We wish that the result is useful for pedagogical purposes.

REFERENCES

Hull, J.C. *Options, Futures, & Other Derivatives.* Edition 7, Pearson, 2009.

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