Asset Pricing in a Pure Exchange Economy with Heterogeneous Investors

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Abstract

In this paper, we provide the first complete solution to the problem of asset pricing in a pure exchange economy with two types of heterogeneous investors: an institutional/retail one with lower/higher risk-aversion. Using a perturbation method with heterogeneity as a small parameter, we solve the equilibrium and obtain analytical approximate formulas for the optimal consumption-sharing rule, pricing function, Sharpe ratio, risk-free rate, stock price and optimal trading strategies. We then analyse the properties of the equilibrium and derive some testable hypotheses, which enhance our understanding on the economics of financial market.

Keywords: Asset pricing; heterogeneous preferences; equilibrium; perturbation methods.

JEL Classifications: G12; D51.

1. Introduction

This paper provides the first complete solution to the problem of asset pricing in a pure exchange economy with two types of heterogeneous investors. Unlike the classical equilibrium model with one representative investor, two agents exchange their goods in the economy due to their heterogeneous risk preferences. Hence, there exist optimal trading strategies and equilibrium asset prices. This equilibrium model has many economic implications, such as modelling competitive market in microeconomics, the evaluation of economic policy in macroeconomics and asset pricing in finance. However, the two-agent equilibrium problem leads to several difficulties in solving the optimal consumption rule and deriving pricing function in a corresponding Arrow-Debreu equilibrium. An explicit solution to this problem is not available yet in the literature. In this paper, using a perturbation method, we are able to solve the two-agent equilibrium model completely. With our analytical approximate solution, we are able to study the impact of the institutional investor in the financial market.

The proportion of U.S. public equities managed by institutions has risen steadily over the past six decades, from about 7 or 8% of market capitalization in 1950, to about 67% in 2010. The large presence of institutional investors has important implications for stock price behaviour. Allen (2001) claims that financial institutions affect asset prices not only because of an agency problem they create but also because of their role in providing liquidity. Institutional investors act as highly specialized investors on behalf of others. They have sophisticated technology to analyse the financial market as risk takers who aggressively invest in the risky asset. To capture this risk-taking feature, we assume one of two types of heterogeneous agents is institutional investor and another one is retail investor, and they have different fractions of initial wealth and heterogeneous risk aversions. In particular, institutional investors as professional risk takers have lower risk aversion. Intuitively, the institutions' fraction and the risk aversion may influence the prices of assets they hold. The purpose of this paper is to explore this mechanism.

Wang (1996) studies the term structure of interest rate in a pure exchange economy with two heterogeneous investors with different constant relative risk aversions, 1 and 0.5 in particular. However, Wang's equilibrium stock price is given as a single integration, which is a solution to an ordinary differential equation (on Page 105). It is not convenient to use his solution in discussing the market equilibrium. Recently, Bhamra and Uppal (2014) consider a general equilibrium in an endowment economy with two types of heterogeneous investors. Each one has heterogeneous belief, risk aversion and time discount rate. Both have "catching up with the Joneses" utility function that is sensitive to habit. They obtain the equilibrium share of consumption by using the Lagrange's theorem. They then derive the price of an asset with a single payoff (named as "spanning assets"), which is written in terms of a hypergeometric function. The equilibrium price of a stock with multiple or continuous dividends is not solved due to the complexity in handling the hypergeometric function in the consumption rule[1].

In this paper, using a perturbation method, we provide the first complete solution to the optimal consumption-sharing rule, pricing function, Sharpe ratio, risk-free rate, stock price (in a long-lived version) and optimal trading strategies in a pure exchange economy with two heterogeneous investors. All perturbation expansions for the equilibrium are simple and tractable. The effects of risk aversion heterogeneity and the size of institutions on the equilibrium are clearly explained by using our approximate solutions.

The remainder of our article is organized as follows. Section 2 presents our model of an exchange economy with heterogeneous agents and the definition of the equilibrium. Section 3 presents our solution to the equilibrium and analyses its properties. Section 4 concludes. Appendix A and B present the homogeneous-investors version of our model and the explicit solution for our maximization problem using Bhamra and Uppal's

[1]If we substitute Bhamra and Uppal's (2014) solutions of our model in Appendix B into the definition of the marginal utility of the representative investor in (11) and then solve the marginal utility, we will find that it is very difficult to explicitly solve the equilibrium price of long-lived stock (12). This is because the marginal utility is with respect to the γ_Ath (or γ_Bth) power of a hypergeometric function.

(2014) method, respectively. Appendix C collects all proofs.

2. Model setup and market equilibrium

2.1 Model setup

We consider a pure-exchange economy with a finite horizon. There are two assets in this economy: one risk-free asset (bond) in zero net supply with risk-free interest rate r_t determined in the equilibrium, and one risky asset (stock) S_t in net supply of one unit. The risky asset produces a consumption good paid as a dividend D_t. The flow of the dividend D_t at time $t \in [0, T]$ follows a geometric Brownian motion,

$$dD_t = D_t \left(\mu dt + \sigma_D dB_t \right), \tag{1}$$

where B_t is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, P)$. The initial value $D_0 > 0$,the drift $\mu \geq 0$ and the volatility $\sigma_D > 0$ are constant.

There are two types of heterogeneous investors, A and B, have constant relative risk aversion (CRRA) preferences with different risk aversion γ_i ($i = A, B$),

$$u_i(c_{i,t}) = e^{-\delta t} \frac{c_{i,t}^{1-\gamma_i}}{1-\gamma_i}. \tag{2}$$

where $\delta > 0$ is the time discount parameter and is the same across two heterogeneous investors. Following Gârleanu and Pedersen (2011) and Rytchkov (2014), by thinking of type A agents as retail investors who are *Averse* to risk, and of type B agents as institutional investors who are *Brave*, we assume risk aversion $\gamma_B < \gamma_A$. Based on following analysis, heterogeneity in investor preferences plays a crucial role in this economy. Type B agents as institutional investors are professional risk takers who prefer to invest in the risky asset.

In this market, heterogeneous investors continuously trade in two assets, bond and stock. Given initial asset of $W_{i,0} > 0$, each type of investors $i(= A, B)$ choose their consumption-trading strategies $\{c_i, (\phi_i, \psi_i)\}$ in consumption, stock and bond, respectively. The wealth process of type i agents, $W_{i,t}$, at time t subject to,

$$dW_{i,t} = \psi_{i,t} r_t dt + \phi_{i,t}[D_t dt + dS_t] - c_{i,t} dt, \tag{3}$$

with initial assets $W_{A,0} = \alpha S_0$ and $W_{B,0} = \beta S_0$ where $\alpha + \beta = 1$ and $\alpha, \beta \in (0, 1)$. The parameters α and β represents the (initial) fraction or size of the retail and institutional investors in the economy, respectively.

Type i investors maximize their expected objective function by choosing consumption-trading strategies $\{c_i, (\phi_i, \psi_i)\}$,

$$\max_{\{c_i,(\phi_i,\psi_i)\}} E\left[\int_0^T e^{-\delta t} \frac{c_{i,t}^{1-\gamma_i}}{1-\gamma_i} dt\right] \quad s.t. \quad \text{Equation (3)}. \tag{4}$$

As $c_{A,t}$ and $c_{B,t}$ are aggregate equilibrium consumption streams of type A and B investors, respectively, the sum of them is less than the total consumption good (paid as a dividend) D_t. In other words, $c_{A,t} + c_{B,t} \leq D_t$ for $t \in [0, T]$.

2.2 Market equilibrium

We follow the definition of a market equilibrium in Wang (1996) and present it as follows:

Definition 1 (market equilibrium (Wang (1996))). *Equilibrium in our economy is defined in a standard way: equilibrium consumption and portfolios strategies* $\{c_i, (\phi_i, \psi_i)\}$ *and the pair of asset prices* $\{S_t, r_t\}$ *are such that type* i *investors maximize their ex-*

pected objective function in (4)*, and markets clear,*

$$\phi_{A,t} + \phi_{B,t} = 1, \tag{5}$$

$$\psi_{A,t} + \psi_{B,t} = 0. \tag{6}$$

In order to solve this equilibrium, following Wang (1996), we first solve the Pareto-optimal allocations and then given Pareto-optimal allocation, an Arrow-Debreu equilibrium can be derived that supports the allocation.

When both types of investors have positive initial wealth $W_{i,0} > 0$, a consumption pair $\{c_A, c_B\}$ is Pareto optimal if and only if there is a constant $\lambda \in (0, 1)$ such that $\{c_A, c_B\}$ solves the problem

$$\max_{c_A, c_B} E_0 \left[\int_0^T e^{-\delta t} \left(\lambda \frac{c_{A,t}^{1-\gamma_A}}{1-\gamma_A} + (1-\lambda) \frac{c_{B,t}^{1-\gamma_B}}{1-\gamma_B} \right) dt \right], \quad s.t. \quad c_{A,t} + c_{B,t} \leq D_t. \tag{7}$$

Here, the parameter λ is the weight of type A investors in the welfare function to be maximized. This maximizing the expected intertemporal welfare function is equivalent to maximizing the welfare function period by period and state by state. Thus, for each period and state, the maximization problem can be changed to

$$\max_{c_{A,t} + c_{B,t} \leq D_t} e^{-\delta t} \left[\lambda \frac{c_{A,t}^{1-\gamma_A}}{1-\gamma_A} + (1-\lambda) \frac{c_{B,t}^{1-\gamma_B}}{1-\gamma_B} \right]. \tag{8}$$

Denoting $b = \frac{1-\lambda}{\lambda} \in (0, +\infty)$ which is the ratio of the weight of type B investors (institutional investors) in the welfare function over the weight of type A investors (retail investors) in the welfare function, the above maximization problem takes the following form:

$$\max_{c_{A,t} + c_{B,t} \leq D_t} e^{-\delta t} \left[\frac{c_{A,t}^{1-\gamma_A}}{1-\gamma_A} + b \frac{c_{B,t}^{1-\gamma_B}}{1-\gamma_B} \right]. \tag{9}$$

Following Wang (1996), we define a representative investor who has the same utility function as in maximization problem (9), that is,

$$u_r(c_{A,t}, c_{B,t}) = e^{-\delta t}\left[\frac{c_{A,t}^{1-\gamma_A}}{1-\gamma_A} + b\frac{c_{B,t}^{1-\gamma_B}}{1-\gamma_B}\right]. \tag{10}$$

In order to define an Arrow-Debreu equilibrium, we assume that given $b \in (0, +\infty)$ and the corresponding Pareto-optimal allocations $\{\widehat{c}_A, \widehat{c}_B\}$ for maximization problem (9), the marginal utility of the representative investor can be defined as

$$\begin{aligned} M_t :=& \frac{\partial u_r(\widehat{c}_{A,t}, \widehat{c}_{B,t})}{\partial D_t} \\ =& e^{-\delta t}\left(\widehat{c}_{A,t}^{-\gamma_A}\frac{\partial \widehat{c}_{A,t}}{\partial D_t} + b\widehat{c}_{B,t}^{-\gamma_B}\frac{\partial \widehat{c}_{B,t}}{\partial D_t}\right). \end{aligned} \tag{11}$$

Definition 2 (Arrow-Debreu equilibrium Wang (1996)). *Given $b \in (0, +\infty)$ and the corresponding Pareto-optimal allocations $\{\widehat{c}_A, \widehat{c}_B\}$ for maximization problem (9), there exists an Arrow-Debreu equilibrium that leads to the same allocation, with the pricing function given by $p_t = M_t/M_0, t \in [0, T]$. Moreover, in this Arrow-Debreu equilibrium, there exists a dynamic implementation in which prices of traded securities are given by*

$$S_t = E_t\left[\int_t^T \left(\frac{M_s}{M_t}\right) D_s ds\right], \quad r_t = -\frac{1}{dt}E_t\left[\frac{dM_t}{M_t}\right]. \tag{12}$$

Investors optimally choose the consumption plan $\{\widehat{c}_A, \widehat{c}_B\}$, financed respectively by budget-feasible trading strategies, and the securities market clears.

3. Main results

3.1 Optimal consumptions

In terms of the maximization problem (9) with heterogeneous risk aversion, its first-order condition becomes $c_{A,t}^{-\gamma_A} = b\left(D_t - c_{A,t}\right)^{-\gamma_B}$. Although Bhamra and Uppal (2014) have employed the Lagrange's theorem to solve the investor A's consumption $c_{A,t}$, their explicit expression is too complex, which is written in terms of a hypergeometric function (see Appendix B). Our target is to analyze the equilibrium asset prices and trading strategies in a pure exchange economy with two heterogeneous investors. Based on this, we use perturbation methods which rely on there being a dimensionless parameter in the problem that is relatively small: $\varepsilon \ll 1$, and assume $\gamma_A = \gamma(1+\varepsilon), \gamma_B = \gamma(1-\varepsilon)$. Here $\varepsilon > 0$ is the risk-averse heterogeneity and γ is mean of risk aversion in the market. Furthermore, we assume γ and δ satisfy $\delta - \mu(1-\gamma) + \frac{1}{2}\sigma_D^2\gamma(1-\gamma) > 0$ to guarantee $S_t > 0$. Then the maximization problem (9) becomes form:

$$\max_{c_{A,t}+c_{B,t} \le D_t} e^{-\delta t} \left[\frac{c_{A,t}^{1-(\gamma+\gamma\varepsilon)}}{1-(\gamma+\gamma\varepsilon)} + b \frac{c_{B,t}^{1-(\gamma-\gamma\varepsilon)}}{1-(\gamma-\gamma\varepsilon)} \right]. \tag{13}$$

The first-order condition derives

$$D_t - c_{A,t} = b^{1/\gamma} \left((D_t - c_{A,t}) c_{A,t} \right)^{\varepsilon} c_{A,t}. \tag{14}$$

By using perturbation methods, the solutions of Equation (14) are given by the following proposition.

Proposition 1 (Pareto-optimal consumption allocations). *Given* $b \in (0, +\infty)$, *the corresponding Pareto-optimal consumption allocations* $\{\widehat{c}_A, \widehat{c}_B\}$ *for maximization problem* (13) *are given by*

$$\widehat{c}_{A,t} = \frac{1}{b^{1/\gamma}+1} D_t - \varepsilon \frac{b^{1/\gamma} h_t}{(b^{1/\gamma}+1)^2} D_t - \varepsilon^2 \frac{b^{1/\gamma}(1-b^{1/\gamma})(h_t^2+2h_t)}{(2b^{1/\gamma}+1)(b^{1/\gamma}+1)^2} D_t + \mathcal{O}(\varepsilon^3). \tag{15}$$

and

$$\widehat{c}_{B,t} = \frac{b^{1/\gamma}}{b^{1/\gamma}+1} D_t + \varepsilon \frac{b^{1/\gamma} h_t}{(b^{1/\gamma}+1)^2} D_t + \varepsilon^2 \frac{b^{1/\gamma}(1-b^{1/\gamma})(h_t^2+2h_t)}{(2b^{1/\gamma}+1)(b^{1/\gamma}+1)^2} D_t + \mathcal{O}(\varepsilon^3). \tag{16}$$

where

$$h_t = \ln\left(\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\right) + 2\ln D_t.$$

Pareto-optimal consumption allocations in Proposition 1 are in second-order. If we remove ε^2 terms, they will become first-order perturbation solutions. In order to verify the accuracy of our solutions in Proposition 1, we compare them with Wang's (1996) exact solutions with particular parameters. Following Wang's (1996) parameters, we set $\gamma = \frac{3}{4}, \varepsilon = \frac{1}{3}$ $(\gamma_A = 1, \gamma_B = \frac{1}{2})$. Moreover, we take $b = 1.7$ as $\beta = 67\%$ in U.S. public equity market in 2010. Without loss of generality, we only compare the consumption of type A investors (retail investors), which is given in Figure 1 [2].

Figure 1 shows the second-order perturbation solution is better than the first-order one, but the first-order perturbation solution is enough to approximate the Wang's (1996) exact solution. The first-order perturbation solution fits Wang's (1996) solution against small dividend value with a good accuracy. Moreover, it is more accurate than Bhamra and Uppal's (2014) solution which keeps 2 terms. With more terms, Bhamra and Uppal's (2014) binomial series expansion converges to Wang's (1996) exact solution. Based on above comparison, we keep all perturbation solutions in first-order throughout this paper.

3.2 Equilibrium

By using the Pareto-optimal consumption allocations, we can easily derive an Arrow-Debreu equilibrium in Definition 2. In this Arrow-Debreu equilibrium, equilibrium asset prices, the pricing function, the Sharpe ratio and optimal trading strategies can be determined.

[2] The optimal consumption for investor A in Wang (1996) is

$$\widehat{c}_{A,t} = \frac{1}{2b^2}(\sqrt{1+4b^2 D_t} - 1),$$

and Bhamra and Uppal's (2014) solution is given in Appendix B.

Proposition 2 (Equilibrium). *Given an Arrow-Debreu equilibrium as defined in Definition 2, there exists a market equilibrium in which the solutions for the equilibrium prices of traded securities are given by*

$$r = \delta + \mu\gamma - \frac{1}{2}\sigma_D^2\gamma(\gamma+1) + \varepsilon\gamma\left(\frac{1}{2}\sigma_D^2(2\gamma+1) - \mu\right)\frac{b^{1/\gamma}-1}{b^{1/\gamma}+1} + \mathcal{O}(\varepsilon^2), \tag{17}$$

$$S_t = \left[\frac{1}{\xi}\left(1 - e^{-\xi(T-t)}\right) + \varepsilon(\mu - \frac{1}{2}\sigma_D^2)\frac{\gamma\left(b^{1/\gamma}-1\right)}{b^{1/\gamma}+1}\frac{1}{\xi}\left(\frac{1}{\xi} - \left(T - t + \frac{1}{\xi}\right)e^{-\xi(T-t)}\right)\right]D_t + \mathcal{O}(\varepsilon^2), \tag{18}$$

and the price-dividend ratio is

$$V_t = \frac{1}{\xi}\left(1 - e^{-\xi(T-t)}\right) + \varepsilon(\mu - \frac{1}{2}\sigma_D^2)\frac{\gamma\left(b^{1/\gamma}-1\right)}{b^{1/\gamma}+1}\frac{1}{\xi}\left(\frac{1}{\xi} - \left(T - t + \frac{1}{\xi}\right)e^{-\xi(T-t)}\right) + \mathcal{O}(\varepsilon^2), \tag{19}$$

where

$$\xi = \delta - \mu(1-\gamma) + \frac{1}{2}\sigma_D^2\gamma(1-\gamma);$$

the marginal utility of the representative investor can be approximated by

$$M_t = e^{-\delta t}\left(\frac{1}{b^{1/\gamma}+1}\right)^{-\gamma} D_t^{-\gamma}(1 + \varepsilon g_t) + \mathcal{O}(\varepsilon^2), \tag{20}$$

where

$$g_t = \frac{\gamma}{b^{1/\gamma}+1}\left((b^{1/\gamma}-1)\ln\frac{D_t}{b^{1/\gamma}+1} + b^{1/\gamma}\ln b^{1/\gamma}\right)$$

and the pricing function $p_t = M_t/M_0$*; the Sharpe ratio is*

$$\theta_t = \left(1 - \varepsilon\frac{b^{1/\gamma}-1}{b^{1/\gamma}+1}\right)\gamma\sigma_D + \mathcal{O}(\varepsilon^2); \tag{21}$$

the solutions for heterogeneous investors' optimal consumption strategies are

$$\widehat{c}_{A,t} = \frac{1}{b^{1/\gamma}+1}D_t - \varepsilon\frac{b^{1/\gamma}h_t}{(b^{1/\gamma}+1)^2}D_t + \mathcal{O}(\varepsilon^2), \tag{22}$$

$$\widehat{c}_{B,t} = \frac{b^{1/\gamma}}{b^{1/\gamma}+1}D_t + \varepsilon\frac{b^{1/\gamma}h_t}{(b^{1/\gamma}+1)^2}D_t + \mathcal{O}(\varepsilon^2). \tag{23}$$

which are financed, respectively, by the following trading strategies:

$$\widehat{\phi}_{A,t} = \frac{1}{b^{1/\gamma}+1} - \varepsilon\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\left(h_t + 2 + 2(\mu - \frac{1}{2}\sigma_D^2)\frac{\frac{1}{\xi} - \left(T-t+\frac{1}{\xi}\right)e^{-\xi(T-t)}}{1-e^{-\xi(T-t)}}\right) + \mathcal{O}(\varepsilon^2), \tag{24}$$

$$\widehat{\phi}_{B,t} = 1 - \widehat{\phi}_{A,t}, \tag{25}$$

$$\widehat{\psi}_{A,t} = 2\varepsilon\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\frac{1}{\xi}\left(1-e^{-\xi(T-t)}\right)D_t + \mathcal{O}(\varepsilon^2), \qquad \widehat{\psi}_{B,t} = -\widehat{\psi}_{A,t}; \tag{26}$$

investors' consumption shares are

$$s_A := \frac{\widehat{c}_{A,t}}{D_t} = \frac{1}{b^{1/\gamma}+1} - \varepsilon\frac{b^{1/\gamma}h_t}{(b^{1/\gamma}+1)^2} + \mathcal{O}(\varepsilon^2), \qquad s_B := \frac{\widehat{c}_{B,t}}{D_t} = \frac{b^{1/\gamma}}{b^{1/\gamma}+1} + \varepsilon\frac{b^{1/\gamma}h_t}{(b^{1/\gamma}+1)^2} + \mathcal{O}(\varepsilon^2); \tag{27}$$

the weight ratio b is restricted by

$$\beta = \frac{b^{1/\gamma}}{b^{1/\gamma}+1} + \varepsilon\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\left(h_0 + 2 + 2(\mu - \frac{1}{2}\sigma_D^2)\frac{\frac{1}{\xi} - \left(T+\frac{1}{\xi}\right)e^{-\xi T}}{1-e^{-\xi T}}\right) + \mathcal{O}(\varepsilon^2), \tag{28}$$

where

$$h_t = \ln\left(\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\right) + 2\ln D_t.$$

Remark 1. Our perturbation solutions for the corresponding Pareto-optimal consumption allocations in Proposition 1 give a good accuracy for reasonable dividends. Firstly, although Wang (1996) obtains exact solutions for Pareto-optimal consumption allocations, the equilibrium stock price is given as a single integration following an ordinary differential equation, which is not convenient to use Wang's solution in discussing the market equilibrium. In addition, substituting Bhamra and Uppal's (2014) solution for consumption strategies in Appendix B into the marginal utility of the representative investor defined by (11), we find the marginal utility is a power function in terms of a hypergeometric function, which is too difficult to solve the price of long-lived stock. In this paper, using a perturbation method, we solve the two-agent equilibrium model completely and provide the first complete solution to the equilibrium including the optimal consumption rule, pricing function, Sharpe ratio, risk-free interest rate, stock price and optimal trading strategies. Finally, this paper combines two factors, the size of institutions and heterogeneity, to study their influence on the equilibrium.

Remark 2. In Proposition 2, the risk-free interest rate is constant. In absence of risk-averse heterogeneity ε, the risk-free interest rate $r = \delta + \mu\gamma - \frac{1}{2}\sigma_D^2\gamma(\gamma+1)$ which is corresponding to benchmark case in Appendix A. If $\left(\mu - \frac{1}{2}\sigma_D^2(2\gamma+1)\right)\frac{b^{1/\gamma}-1}{b^{1/\gamma}+1} > 0$, r decreases with the risk-averse heterogeneity in the economy, and vice versa. One situation is that fixing $\mu - \frac{1}{2}\sigma_D^2(2\gamma+1) > 0$ and setting $b > 1$, r decreases with ε. In other words, in a financial market with large presence of institutional investors, higher risk-averse heterogeneity reduces risk-free interest rate. $\mu - \frac{1}{2}\sigma_D^2(2\gamma+1) > 0$ fixed, r decreases with the weight ratio b in the economy.

Remark 3. The price-dividend ratio (the stock price) is time-varying, and its volatility is constant and equals σ_D, from Equation (18). We assume $\mu > \frac{1}{2}\sigma_D^2$, which means that the expected growth rate of log dividend in (1), $\mu - \frac{1}{2}\sigma_D^2$, is positive. Then, if the weight ratio $b > 1$ (which mean institutional investors dominate the financial market), the price-dividend ratio increases with the risk-averse heterogeneity ε, and vice versa. In addition, the weight ratio always pushes up the price-dividend ratio. Because the size of institutions determines the weight ratio, the weight ratio can be regarded as a measure of the size, and the price-dividend ratio is good measure for P/E ratio. Thus, we can understand it as the larger size of institutions (or firms) derives higher P/E ratio.

Remark 4. The Sharpe ratio is constant in Equation (21). In homogeneous case, the Sharpe ratio $\theta = \gamma\sigma_D$ which is corresponding to benchmark case in Appendix A. If $b > 1$, the Sharpe ratio decreases with the risk-averse heterogeneity, and vice versa. Furthermore, the Sharpe ratio decreases with the weight ratio b in the economy.

Remark 5. Given $h_t+2+2(\mu-\frac{1}{2}\sigma_D^2)\frac{\frac{1}{\xi}-\left(T-t+\frac{1}{\xi}\right)e^{-\xi(T-t)}}{1-e^{-\xi(T-t)}} > 0$, the stock investment of institutional investors, $\widehat{\phi}_{B,t}$, goes up with increasing of both the risk-averse heterogeneity and the weight ratio in the economy. The risk-free asset investment of institutional investors is always negative for any $\varepsilon > 0$. It means that institutional investors are always levered. Moreover, higher risk-averse heterogeneity brings higher leverage (borrowing) for institutional investors from Equation (26).

Remark 6. Investors' optimal consumption shares are stochastic and with respect to h_t. If $h_t < 0$ (b is extreme large), institutional investors' consumption share s_B decreases with the risk-averse heterogeneity, and vice versa. Obviously, from Equation (27), institutional investors' consumption share increases with the weight ratio.

Remark 7. The weight ratio b is restricted by Equation (28) which reveals the relationship among β, ε and b. If $b = 1, \varepsilon > 0$, then $\beta > 0.5$. In other world, if $\beta = 0.5, \varepsilon > 0$, then $b > 1$. Given $\beta \in (0, 1)$, b decreases with the risk-averse heterogeneity ε. Obviously, b increases with β. All solutions in Proposition 2 are related to the weight ratio b which is dominated by the size of institutions β. This is consistent with the empirical evidence that financial institutions matter for asset pricing (Allen (2001); Blume and Keim (2012)). In this paper, we argue that the risk-averse heterogeneity affects asset pricing as well. But there is little empirical literature to document this statement as the data of the risk-averse heterogeneity is not available.

3.3 Numerical analysis on the equilibrium

In this subsection, we are going to analyze equilibrium effects of the size of institutions β and the risk-averse heterogeneity ε on asset prices and optimal consumption strategies to understand the economics in financial market. In term of parameters in this paper, we set $D_0 = 1, \mu = 0.02, \sigma_D = 0.01, \gamma = 3/4, \delta = 0.01, T = 100, t = 20, D_t = 2.72$.

3.3.1 Effects of the size of institutions β. In benchmark homogeneous case (see Appendix A), it shows that the risk-free interest rate and the Sharpe ratio are constant and not related to the size of institutions. It means that if retail and institutional investors are homogeneous, the risk-free interest rate and the Sharpe ratio are only influenced by their common time discount parameter, risk aversion and parameters of the dividend. Moreover, the price-dividend ratio in benchmark case is time varying. If the model is in a infinite horizon or at a fixed time (e.g. $t = 20$), the price-dividend ratio will become a constant. The weight ratio $b = \left(\frac{\beta}{1-\beta}\right)^{\gamma}$ which is a monotonically increasing function with respect to $\beta \in (0, 1)$. The results are presented in Figure 2 with the dashed line.

In an economy with two heterogeneous investors, Panel A shows that the larger size of institutions (e.g. $\beta = 0.8$) makes them have higher weight in the welfare function than the smaller size of institutions (e.g. $\beta = 0.2$). Panels B, C and D illustrate that with increasing of the size of institutions β, the equilibrium Sharpe ratio θ and risk-free interest rate r decrease but the price-dividend ratio V goes up. Combined with Figure 3, Panel D and F, with the larger size of institutions, the institutions demand riskier portfolios. However, since the risky stock is in fixed supply, it must become less attractive in the presence of institutions to clear markets. So, the market Sharpe ratio decreases more heavily with more institutions in the economy. Because institutions short the risk-free asset to push up the supply of bonds, the price of risk-free assets declines. Those effects are similar to that in Basak and Pavlova (2013).

In benchmark homogeneous case, the optimal consumption share of institutional investors and holdings of the stock equals β, and this relationship is an identity function. Furthermore, holdings of the risk-free asset keep zero. However, in heterogeneous case, the relationship between holdings of the stock equals β and the size of institution β is nonlinear. The largest difference is that the plot of institutional investors shorting the risk-free asset (borrowing money) is bell-shaped. As Basak and Pavlova (2013) claim that at first, with increasing of the size of institutional investors, they are able to borrow more money to invest in the risky asset. At a certain point, as the institutions become larger, the size of the retail investors shrinks, therefore, borrowing of institutional investors reduces in the economy. It is a very important illustration of how leverage in the economy depends on the size of the institutions.

Testable hypotheses on effects of the size of institutions: In an Arrow-Debreu equilibrium with heterogeneous institutional investors,

(i) the weight ratio b (which is the ratio of the weight of institutional investors over the weight of retail investors in the welfare function) increases with the size of institutional investors , β, in the economy;

(ii) the Sharpe ratio θ and the risk-free interest rate r decrease with the size of institutional investors;

(iii) the price-dividend ratio V increases with the size of institutional investors β;

(iv) the consumption share of institutional investors s_B increases with β;

(v) the stock investment of institutional investors $\widehat{\phi}_B$ increases with the size of institutional investors;

(vi) for $\beta \in (0, 1)$, the institutional investor is always levered, $\psi_B < 0$.

Basak and Pavlova (2013) investigate how the size of institutions influences equilibrium asset prices based on a dynamic general equilibrium model with two classes of investors: retail investors and institutional investors. The distinction between these is that the utility function of institutional investors is sensitive to the value of stock index but that of retail investors not. In contrast to the definition of retail investors and institutional investors in Basak and Pavlova (2013), we follow Gârleanu and Pedersen

(2011) and Rytchkov (2014) and define that who have less risk aversion are institutional investors (e.g. hedge funds) and who have relatively higher risk aversion are retail investors. It means that institutional investors are professional risk takers that aggressively invest in the risky asset. It is corresponding to our finding that increasing of the size of institutions pushes up the stock price because of their higher demands for stocks. We study the same issue in the same way as Basak and Pavlova (2013). Like Basak and Pavlova (2013), our model illustrates that an increase in the size of institutions leads to the equilibrium Sharpe ratio decreasing but the price-dividend ratio going up, and the plot of institutional investors shorting the risk-free asset is bell-shaped. In addition, this paper combines another factor, heterogeneity of risk aversion, with the size of institutions to explore the economic mechanism how these two factors simultaneously affect the equilibrium.

3.3.2 Effects of the risk-averse heterogeneity ε. In Figure 4, the weight ratio b falls with increasing of institutional investors' risk aversion (Figure 4, Panel A), but there is no effects of the risk-averse heterogeneity on the weight ratio b when the size of the institutional investors is zero ($\beta = 0$) and one ($\beta = 1$). The effects on the Sharpe ratio, risk-free interest rate and price-dividend ratio depend on the size of institutions. Generally speaking, compared with price-dividend ratio, the Sharpe ratio and risk-free interest rate have opposite response to the risk-averse heterogeneity for the same size of institutions.

Corresponding to Figure 3 (Panel A-D), the consumption share and the risky asset investment of the two types of heterogeneous investors have a weak response to the risk-averse heterogeneity, but the investment of the risky asset slightly goes up with the risk-averse heterogeneity. Except for homogeneous case ($\beta = 0, 1$), the higher risk-averse heterogeneity forces institutional investor borrowing more money or having higher leverage.

Testable hypotheses on effects of the risk-averse heterogeneity: In an Arrow-

Debreu equilibrium with presence of institutional investors,

(i) the Sharpe ratio θ and the risk-free interest rate r decrease for institutional investors with a large size (e.g. $\beta = 0.75$) but increase for those with a small size (e.g. $\beta = 0.25$) with the increase of the risk-averse heterogeneity ε;

(ii) the price-dividend ratio V increases for institutional investors with a large size (e.g. $\beta = 0.75$) but decreases for those with a small size (e.g. $\beta = 0.25$) with the increase of the risk-averse heterogeneity ε;

(iii) the weight ratio b and institutional investors' consumption share s_B slightly go down while stock investment of institutional investors $\widehat{\phi}_B$ slightly goes up with the increase of the risk-averse heterogeneity ε;

(iv) for $\beta \in (0, 1)$, the institutional investors are always levered, and the leverage rises with the ε in the economy.

Based on above analysis, our perturbation solutions for the equilibrium model in a pure exchange economy can perfectly explain the effects of the size of institutions and the risk-averse heterogeneity on the Sharpe ratio, the risk-free interest rate, the stock price and optimal trading strategies for institutional and retail investors. In contrast to Wang (1996); Bhamra and Uppal (2014), this paper explicitly and completely solves investors' optimal trading strategies and the risky asset (stock) price. In addition, Bhamra and Uppal (2014) assume the weight ratio in the welfare function equals the ratio of initial wealth. According to Basak (2005); Buraschi et al. (2014), however, the weight ratio in the welfare function should be the ratio of the Lagrange multiplier on the static budget constraint of each type of investors.

To summarize, following a standard framework in Wang (1996), this paper reveals the mechanism of how the size of institutions (the initial wealth) and the risk-averse heterogeneity affect on the weight ratio and the equilibrium.

4. Conclusion

In this paper, we study a pure exchange economy in which there are two types of investors, each with CRRA utility. One type of investors with lower risk aversion models institutional investors and another one with higher risk aversion models retail investors. This paper presents theoretical analysis and a complete solution to the equilibrium in an exchange economy with these two heterogeneous investors.

Our paper is the first to completely solve for the equilibrium in this economy and to identify the optimal consumption-sharing rule, pricing function, Sharpe ratio, risk-free rate, stock price and optimal trading strategies for each type of investors. Our solutions are written as functions of the size of institutions in a perturbation form with risk-aversion heterogeneity as a small parameter. Numerical experiments show that our solution is accurate. It is very convenient to use our solutions as they are in a closed-form.

Taking advantage of tractability of our solutions, we analyse the effects of the size of institutions and the risk-aversion heterogeneity on the equilibrium. We agree with Basak and Pavlova (2013) on the effect of the size of institutions, and with Bhamra and Uppal (2014) on the effect of the risk-aversion heterogeneity. These effects help us to better understand the economics of financial market.

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Appendix A: An exchange economy with two homogeneous investors

In this benchmark homogeneous case, we set $\gamma_A = \gamma_B = \gamma$ and then the maximization problem in (9) becomes,

$$\max_{c_{A,t}+c_{B,t}\leq D_t} e^{-\delta t}\left[\frac{c_{A,t}^{1-\gamma}}{1-\gamma} + b\frac{c_{B,t}^{1-\gamma}}{1-\gamma}\right].$$

The first order condition produces

$$\frac{D_t - c_{A,t}}{c_{A,t}} = b^{1/\gamma}.$$

Then, we can easily solve above linear equation and obtain the following results.

Pareto-optimal allocations: Given $b \in (0, +\infty)$, Pareto-optimal allocations of two homogeneous investors with the same risk aversion γ are

$$\widehat{c}_{A,t} = \frac{1}{b^{1/\gamma}+1} D_t,$$

$$\widehat{c}_{B,t} = D_t \widehat{c}_{A,t} = \frac{b^{1/\gamma}}{b^{1/\gamma}+1} D_t.$$

Furthermore, the marginal utility of the representative investor is

$$M_t = e^{-\delta t}\left(\frac{1}{b^{1/\gamma}+1}\right)^{-\gamma} D_t^{-\gamma}.$$

In this benchmark case, as b is restricted by the budget constraint $W_A(0) = \alpha S(0)$, we have $b = \left(\frac{1-\alpha}{\alpha}\right)^{\gamma}$. Since $\alpha = 1 - \beta$, the marginal utility is solved as

$$M_t = e^{-\delta t}(1-\beta)^{-\gamma} D_t^{-\gamma}.$$

Its stochastic differential equation process is

$$\frac{dM_t}{M_t} = -[\delta + \mu\gamma - \frac{1}{2}\sigma_D^2\gamma(\gamma+1)]dt - \gamma\sigma_D dB_t,$$

and the Sharpe ratio is

$$\theta := \gamma\sigma_D.$$

Equilibrium: Given an Arrow-Debreu equilibrium with two homogeneous investors with same risk aversion γ, there exists a market equilibrium in which the equilibrium prices of traded securities are given by

$$\begin{aligned} r &= \delta + \mu\gamma - \frac{1}{2}\sigma_D^2\gamma(\gamma+1), \\ S_t &= \frac{D_t}{\xi}\left(1 - e^{-\xi(T-t)}\right), \end{aligned}$$

and the price-dividend ratio is

$$V_t := \frac{S}{D} = \frac{1}{\xi}\left(1 - e^{-\xi(T-t)}\right),$$

where $\xi := \delta - \mu(1-\gamma) + \frac{1}{2}\sigma_D^2\gamma(1-\gamma)$; homogeneous investors' optimal consumption strategies are

$$\widehat{c}_{A,t} = \alpha D_t, \qquad \widehat{c}_{B,t} = \beta D_t,$$

which are financed, respectively, by the following trading strategies:

$$\begin{aligned} \widehat{\phi}_{A,t} &= \alpha, \qquad \widehat{\phi}_{B,t} = \beta, \\ \widehat{\psi}_{A,t} &= 0, \qquad \widehat{\psi}_{B,t} = 0. \end{aligned}$$

And investors' consumption shares are

$$s_A := \frac{\widehat{c}_{A,t}}{D_t} = \alpha, \qquad s_B := \frac{\widehat{c}_{B,t}}{D_t} = \beta.$$

Appendix B: The solution for the maximization problem (9) by using Bhamra and Uppal's (2014) method

Here, we use Bhamra and Uppal's (2014) method to solve the maximization problem (9). The first-order condition for the maximization problem (9) is

$$c_{A,t}^{-\gamma_A} = bc_{B,t}^{-\gamma_B}$$

By denoting $s_{A,t} := \frac{\widehat{c}_{A,t}}{D_t}$ and $s_{B,t} := \frac{\widehat{c}_{B,t}}{D_t}$, then,

$$D_t^{-\gamma_A} s_{A,t}^{-\gamma_A} = bD_t^{-\gamma_B} s_{B,t}^{-\gamma_B}.$$

It can be rewritten as

$$s_{B,t} = A_t s_{A,t}^{\eta},$$

where $A_t = \left(\frac{bD_t^{-\gamma_B}}{D_t^{-\gamma_A}}\right)^{\frac{1}{\gamma_B}} = b^{\frac{1}{\gamma_B}} D_t^{\frac{\gamma_A - \gamma_B}{\gamma_B}}$, $\eta = \frac{\gamma_A}{\gamma_B}$ and $s_{A,t} + s_{B,t} = 1$. In our paper, we assume $\gamma_B < \gamma_A$.

The solutions for above equation are

$$s_{B,t} = \begin{cases} 1 + \sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^n}{n} \begin{pmatrix} \frac{n}{\eta} \\ n-1 \end{pmatrix}, & A_t > \overline{R}, \\ -\sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \begin{pmatrix} n\eta \\ n-1 \end{pmatrix}, & A_t < \overline{R}, \end{cases}$$

and

$$s_{A,t} = \begin{cases} -\sum_{n=1}^{\infty} \frac{\left(-A_t^{-\frac{1}{\eta}}\right)^n}{n} \begin{pmatrix} \frac{n}{\eta} \\ n-1 \end{pmatrix}, & A_t > \overline{R}, \\ 1 + \sum_{n=1}^{\infty} \frac{(-A_t)^n}{n} \begin{pmatrix} n\eta \\ n-1 \end{pmatrix}, & A_t < \overline{R}, \end{cases}$$

where $\overline{R} = \frac{(\eta-1)^{\eta-1}}{\eta^\eta}$; for $z \in \mathbb{C}$ and $k \in \mathbb{N}$, $\begin{pmatrix} z \\ k \end{pmatrix} = \Pi_{j=1}^{k} \frac{z-k+j}{j}$ is the generalized binomial coefficient and for $z, k \in \mathbb{R}^+$, $\begin{pmatrix} z \\ k \end{pmatrix} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\Gamma(k+1)}$, the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$.

Finally, we have the binomial series expressions for the maximization problem (9),

$$\widehat{c}_{A,t} = \begin{cases} D_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(b^{-1} D_t^{\gamma_B-\gamma_A}\right)^{\frac{n}{\gamma_A}} \begin{pmatrix} n\frac{\gamma_B}{\gamma_A} \\ n-1 \end{pmatrix}, & D_t > b^{\frac{1}{\gamma_B-\gamma_A}} \frac{\gamma_A-\gamma_B}{\gamma_B} \left(\frac{\gamma_A}{\gamma_B}\right)^{\frac{\gamma_A}{\gamma_B-\gamma_A}}, \\ D_t - D_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(b D_t^{\gamma_A-\gamma_B}\right)^{\frac{n}{\gamma_B}} \begin{pmatrix} n\frac{\gamma_A}{\gamma_B} \\ n-1 \end{pmatrix}, & D_t < b^{\frac{1}{\gamma_B-\gamma_A}} \frac{\gamma_A-\gamma_B}{\gamma_B} \left(\frac{\gamma_A}{\gamma_B}\right)^{\frac{\gamma_A}{\gamma_B-\gamma_A}}, \end{cases}$$

and

$$\widehat{c}_{B,t} = \begin{cases} D_t - D_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(b^{-1} D_t^{\gamma_B-\gamma_A}\right)^{\frac{n}{\gamma_A}} \begin{pmatrix} n\frac{\gamma_B}{\gamma_A} \\ n-1 \end{pmatrix}, & D_t > b^{\frac{1}{\gamma_B-\gamma_A}} \frac{\gamma_A-\gamma_B}{\gamma_B} \left(\frac{\gamma_A}{\gamma_B}\right)^{\frac{\gamma_A}{\gamma_B-\gamma_A}}, \\ D_t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(b D_t^{\gamma_A-\gamma_B}\right)^{\frac{n}{\gamma_B}} \begin{pmatrix} n\frac{\gamma_A}{\gamma_B} \\ n-1 \end{pmatrix}, & D_t < b^{\frac{1}{\gamma_B-\gamma_A}} \frac{\gamma_A-\gamma_B}{\gamma_B} \left(\frac{\gamma_A}{\gamma_B}\right)^{\frac{\gamma_A}{\gamma_B-\gamma_A}}. \end{cases}$$

Appendix C: Proofs for Propositions

1. Proof of Proposition 1: According to perturbation methods, we denote $f(D_t;\varepsilon) := \widehat{c}_{A,t}$. For a small ε, $f(D_t;\varepsilon)$ can be approximated in second-order by,

$$f(D_t;\varepsilon) = f_{0,t} + f_{1,t}\varepsilon + f_{2,t}\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Substituting it into (14), we get,

$$\begin{aligned} & D_t - f_{0,t} - f_{1,t}\varepsilon - f_{2,t}\varepsilon^2 \\ =& b^{1/\gamma}\left[f_{0,t} + \left(f_{0,t}\ln\left[\left(D_t - f_{0,t}\right)f_{0,t}\right] + f_{1,t}\right)\varepsilon + f_{0,t}\ln^2\left[\left(D_t - f_{0,t}\right)f_{0,t}\right]\varepsilon^2 \right. \\ & \left. + \frac{2\left(-f_{1,t}f_{0,t} + (D_t - f_{0,t})f_{1,t}\right)}{D_t - f_{0,t}}\varepsilon^2 + 2f_{1,t}\ln\left[\left(D_t - f_{0,t}\right)f_{0,t}\right] + 2f_{2,t}\varepsilon^2\right] + \mathcal{O}(\varepsilon^3). \end{aligned}$$

We can collect powers of

$$\begin{aligned} \mathcal{O}(\varepsilon^0): \quad & D_t - f_{0,t} = b^{1/\gamma}f_{0,t}, \\ \mathcal{O}(\varepsilon^1): \quad & -f_{1,t} = b^{1/\gamma}\left(f_{0,t}\ln\left[\left(D_t - f_{0,t}\right)f_{0,t}\right] + f_{1,t}\right), \\ \mathcal{O}(\varepsilon^2): \quad & -f_{2,t}\varepsilon^2 = b^{1/\gamma}\left(f_{0,t}\ln^2\left[\left(D_t - f_{0,t}\right)f_{0,t}\right] + \frac{2\left(-f_{1,t}f_{0,t} + (D_t - f_{0,t})f_{1,t}\right)}{D_t - f_{0,t}} \right. \\ & \left. + 2f_{1,t}\ln\left[\left(D_t - f_{0,t}\right)f_{0,t}\right] + 2f_{2,t}\right). \end{aligned}$$

Now we simply solve at each order,

$$f_{0,t} = \frac{1}{b^{1/\gamma}+1}D_t, \quad f_{1,t} = -\frac{b^{1/\gamma}h_t f_{0,t}}{b^{1/\gamma}+1} = -\frac{b^{1/\gamma}h_t}{(b^{1/\gamma}+1)^2}D_t,$$

and

$$f_{2,t} = \frac{b^{1/\gamma}(1-b^{1/\gamma})(h_t^2+2h_t)}{(2b^{1/\gamma}+1)(b^{1/\gamma}+1)^2}D_t.$$

where $h_t = \ln\left[\left(D_t - f_{0,t}\right)f_{0,t}\right] = \ln\left(\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\right) + 2\ln D_t$.

2. Proof of Proposition 2: Following the definition of the marginal utility of the representative investor in (11), the marginal utility in first-order approximation is

$$\begin{aligned}
M_t &= e^{-\delta t}\widehat{c}_{A,t}^{-\gamma(1+\varepsilon)}\frac{\partial \widehat{c}_{A,t}}{\partial D_t} + e^{-\delta t} b(\widehat{c}_{B,t})^{-\gamma(1-\varepsilon)}\frac{\partial \widehat{c}_{B,t}}{\partial D_t} \\
&= e^{-\delta t}\left[\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}h_t\right)\frac{1}{b^{1/\gamma}+1}D_t\right]^{-\gamma-\gamma\varepsilon}\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}(h_t+2)\right)\frac{1}{b^{1/\gamma}+1} \\
&+ e^{-\delta t} b\left[\left(1+\varepsilon\frac{1}{b^{1/\gamma}+1}h_t\right)\frac{b^{1/\gamma}}{b^{1/\gamma}+1}D_t\right]^{-\gamma+\gamma\varepsilon}\left(1+\varepsilon\frac{1}{b^{1/\gamma}+1}(h_t+2)\right)\frac{b^{1/\gamma}}{b^{1/\gamma}+1} \\
&+ \mathcal{O}(\varepsilon^2) \\
&= e^{-\delta t}\left(\frac{1}{b^{1/\gamma}+1}\right)^{-\gamma} D_t^{-\gamma}\left[1+\varepsilon\frac{\gamma}{b^{1/\gamma}+1}\left((b^{1/\gamma}-1)\ln\frac{D_t}{b^{1/\gamma}+1}+b^{1/\gamma}\ln b^{1/\gamma}\right)\right] \\
&+ \mathcal{O}(\varepsilon^2).
\end{aligned}$$

Thus, the marginal utility of the representative investor can be approximated in first-order by

$$M_t = e^{-\delta t}\left(\frac{1}{b^{1/\gamma}+1}\right)^{-\gamma} D_t^{-\gamma}(1+\varepsilon g_t) + \mathcal{O}(\varepsilon^2).$$

where $g_t = g_1 \ln D_t + g_0$, $g_1 = \frac{\gamma(b^{1/\gamma}-1)}{b^{1/\gamma}+1}$ and $g_0 = \frac{\gamma}{b^{1/\gamma}+1}\left(b^{1/\gamma}\ln b^{1/\gamma} - (b^{1/\gamma}-1)\ln(b^{1/\gamma}+1)\right)$. Then, the stochastic differential equation process of the marginal utility is

$$\begin{aligned}
\frac{dM_t}{M_t} &= -\left[\delta + \mu\gamma - \frac{1}{2}\sigma_D^2\gamma(\gamma+1) + \varepsilon g_1\left(\frac{1}{2}\sigma_D^2(2\gamma+1)-\mu\right)\right]dt \\
&\quad - (\gamma - \varepsilon g_1)\sigma_D dB_t,
\end{aligned}$$

and the first-order perturbation solution for Sharpe ratio is

$$\theta_t = (\gamma - \varepsilon g_1)\sigma_D + \mathcal{O}(\varepsilon^2).$$

Since $D_s = D_t e^{(\mu - \frac{1}{2}\sigma_D^2)(s-t)+\sigma_D(B_s - B_t)}$, then $\ln D_s = \ln D_t + (\mu - \frac{1}{2}\sigma_D^2)(s-t) + \sigma_D(B_s - B_t)$, $E_t[D_s] = D_t e^{\mu(s-t)}$ and $E_t[\ln D_s] = \ln D_t + (\mu - \frac{1}{2}\sigma_D^2)(s-t)$. Thus, the stock price is

$$\begin{aligned}
S_t &= E_t\left[\int_t^T \left(\frac{M_s}{M_t}\right) D_s ds\right] \\
&= \frac{1}{D_t^{-\gamma}(1+\varepsilon g_t)} E_t\left[\int_t^T e^{-\delta(s-t)} D_s^{1-\gamma}(1+\varepsilon g_s) ds\right] + \mathcal{O}(\varepsilon^2) \\
&= D_t \frac{1}{(1+\varepsilon g_t)} \int_t^T e^{-\xi(s-t)} \left[1 + \varepsilon g_t + \varepsilon g_1(\mu - \frac{1}{2}\sigma_D^2)(s-t)\right] ds + \mathcal{O}(\varepsilon^2) \\
&= D_t \int_t^T e^{-\xi(s-t)} ds + \varepsilon D_t g_1(\mu - \frac{1}{2}\sigma_D^2) \int_t^T e^{-\xi(s-t)}(s-t) ds + \mathcal{O}(\varepsilon^2) \\
&= \left[\frac{1}{\xi}\left(1 - e^{-\xi(T-t)}\right) + \varepsilon g_1(\mu - \frac{1}{2}\sigma_D^2)\frac{1}{\xi}\left(\frac{1}{\xi} - \left(T - t + \frac{1}{\xi}\right) e^{-\xi(T-t)}\right)\right] D_t + \mathcal{O}(\varepsilon^2). \\
&= \left[\frac{1}{\xi}\left(1 - e^{-\xi(T-t)}\right) + \varepsilon(\mu - \frac{1}{2}\sigma_D^2)\frac{\gamma\left(b^{1/\gamma} - 1\right)}{b^{1/\gamma} + 1}\frac{1}{\xi}\left(\frac{1}{\xi} - \left(T - t + \frac{1}{\xi}\right) e^{-\xi(T-t)}\right)\right] D_t \\
&\quad + \mathcal{O}(\varepsilon^2).
\end{aligned}$$

The wealth of investor A is

$$
\begin{aligned}
W_{A,t} &= E_t\left[\int_t^T \left(\frac{M_s}{M_t}\right)\widehat{c}_{A,s}ds\right] \\
&= \frac{1}{b^{1/\gamma}+1}\frac{1}{D_t^{-\gamma}(1+\varepsilon g_t)}E_t\left[\int_t^T e^{-\delta(s-t)}D_s^{1-\gamma}(1+\varepsilon g_s)\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}h_s\right)ds\right] \\
&+ \mathcal{O}(\varepsilon^2) \\
&= \frac{D_t}{b^{1/\gamma}+1}\frac{1}{(1+\varepsilon g_t)}\int_t^T e^{-\xi(s-t)}\left[\left(1+\varepsilon g_t+\varepsilon\gamma\frac{b^{1/\gamma}-1}{b^{1/\gamma}+1}(\mu-\frac{1}{2}\sigma_D^2)(s-t)\right)\right. \\
&\cdot\left.\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}h_t-2\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}(\mu-\frac{1}{2}\sigma_D^2)(s-t)\right)\right]ds+\mathcal{O}(\varepsilon^2) \\
&= \frac{D_t}{b^{1/\gamma}+1}\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}h_t\right)\int_t^T e^{-\xi(s-t)}ds \\
&+ \varepsilon(\mu-\frac{1}{2}\sigma_D^2)\frac{D_t}{b^{1/\gamma}+1}\frac{\gamma\left(b^{1/\gamma}-1\right)-2b^{1/\gamma}}{b^{1/\gamma}+1}\int_t^T e^{-\xi(s-t)}(s-t)ds+\mathcal{O}(\varepsilon^2) \\
&= \frac{D_t}{b^{1/\gamma}+1}\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}h_t\right)\frac{1}{\xi}\left(1-e^{-\xi(T-t)}\right) \\
&+ \varepsilon(\mu-\frac{1}{2}\sigma_D^2)\frac{D_t}{b^{1/\gamma}+1}\frac{\gamma\left(b^{1/\gamma}-1\right)-2b^{1/\gamma}}{b^{1/\gamma}+1}\frac{1}{\xi}\left(\frac{1}{\xi}-\left(T-t+\frac{1}{\xi}\right)e^{-\xi(T-t)}\right)+\mathcal{O}(\varepsilon^2)
\end{aligned}
$$

Then

$$
\begin{aligned}
&\widehat{\phi}_{A,t} = \frac{dW_{A,t}}{dS_t} = \frac{dW_{A,t}}{\frac{dS_t}{dD_t}dD_t} = \frac{dW_{A,t}}{dD_t}\Big/\frac{dS_t}{dD_t} \\
=&\frac{\frac{1}{b^{1/\gamma}+1}\left(1-\varepsilon\frac{b^{1/\gamma}}{b^{1/\gamma}+1}(h_t+2)\right)\frac{1}{\xi}\left(1-e^{-\xi(T-t)}\right)+\varepsilon(\mu-\frac{1}{2}\sigma_D^2)\frac{\gamma\left(b^{1/\gamma}-1\right)-2b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\frac{1}{\xi}\left(\frac{1}{\xi}-\left(T-t+\frac{1}{\xi}\right)e^{-\xi(T-t)}\right)}{\frac{1}{\xi}\left(1-e^{-\xi(T-t)}\right)+\varepsilon(\mu-\frac{1}{2}\sigma_D^2)\frac{\gamma\left(b^{1/\gamma}-1\right)}{b^{1/\gamma}+1}\frac{1}{\xi}\left(\frac{1}{\xi}-\left(T-t+\frac{1}{\xi}\right)e^{-\xi(T-t)}\right)} \\
&+\mathcal{O}(\varepsilon^2) \\
=&\frac{1}{b^{1/\gamma}+1}-\varepsilon\left[\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}(h_t+2)+\frac{(\mu-\frac{1}{2}\sigma_D^2)\frac{2b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\left(\frac{1}{\xi}-\left(T-t+\frac{1}{\xi}\right)e^{-\xi(T-t)}\right)}{1-e^{-\xi(T-t)}}\right]+\mathcal{O}(\varepsilon^2),
\end{aligned}
$$

and

$$\widehat{\psi}_{A,t} = W_{A,t} - \widehat{\phi}_{A,t} S_t = 2\varepsilon \frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2} \frac{1}{\xi} \left(1 - e^{-\xi(T-t)}\right) D_t + \mathcal{O}(\varepsilon^2).$$

Therefore,

$$\begin{aligned} \widehat{\phi}_{B,t} &= 1 - \widehat{\phi}_{A,t} \\ \widehat{\psi}_{B,t} &= -\widehat{\psi}_{A,t}. \end{aligned}$$

Using initial wealth of investor A, $W_{A,0} = \alpha S_0$, the weight ratio b is restricted by

$$1 - \beta = \alpha = \widehat{\phi}_{A,0} = \frac{1}{b^{1/\gamma}+1} - \varepsilon \left[\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}(h_0+2) + \frac{(\mu - \frac{1}{2}\sigma_D^2)\frac{2b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\left(\frac{1}{\xi} - \left(T + \frac{1}{\xi}\right) e^{-\xi T}\right)}{1 - e^{-\xi T}} \right],$$

where $h_0 = \ln\left(\frac{b^{1/\gamma}}{(b^{1/\gamma}+1)^2}\right) + 2\ln D_0$.

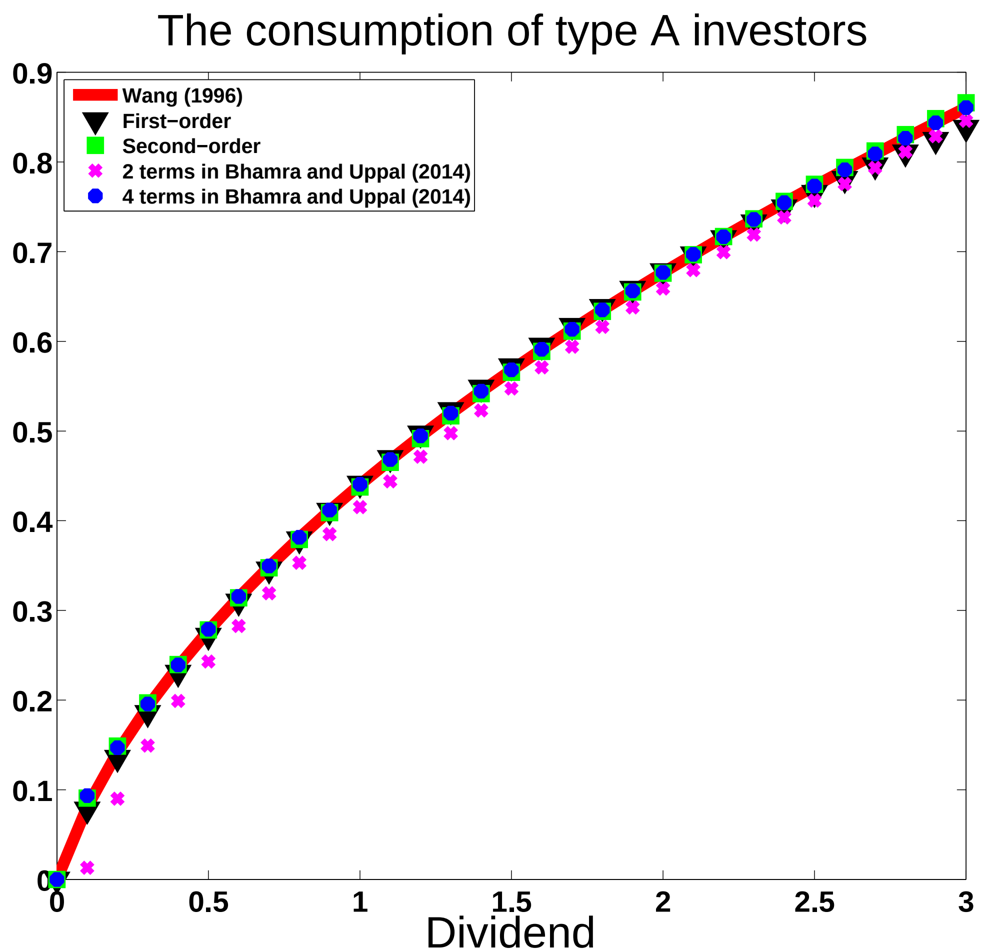

Figure 1: The consumption of type A investors. This figure shows the value of type A investors' consumption by using first-order and second-order approximations in our paper and that of Wang's (1996) and Bhamra and Uppal's (2014). The model parameters are as follows: $\gamma = \frac{3}{4}, \varepsilon = \frac{1}{3}$ and $b = 1.7$.

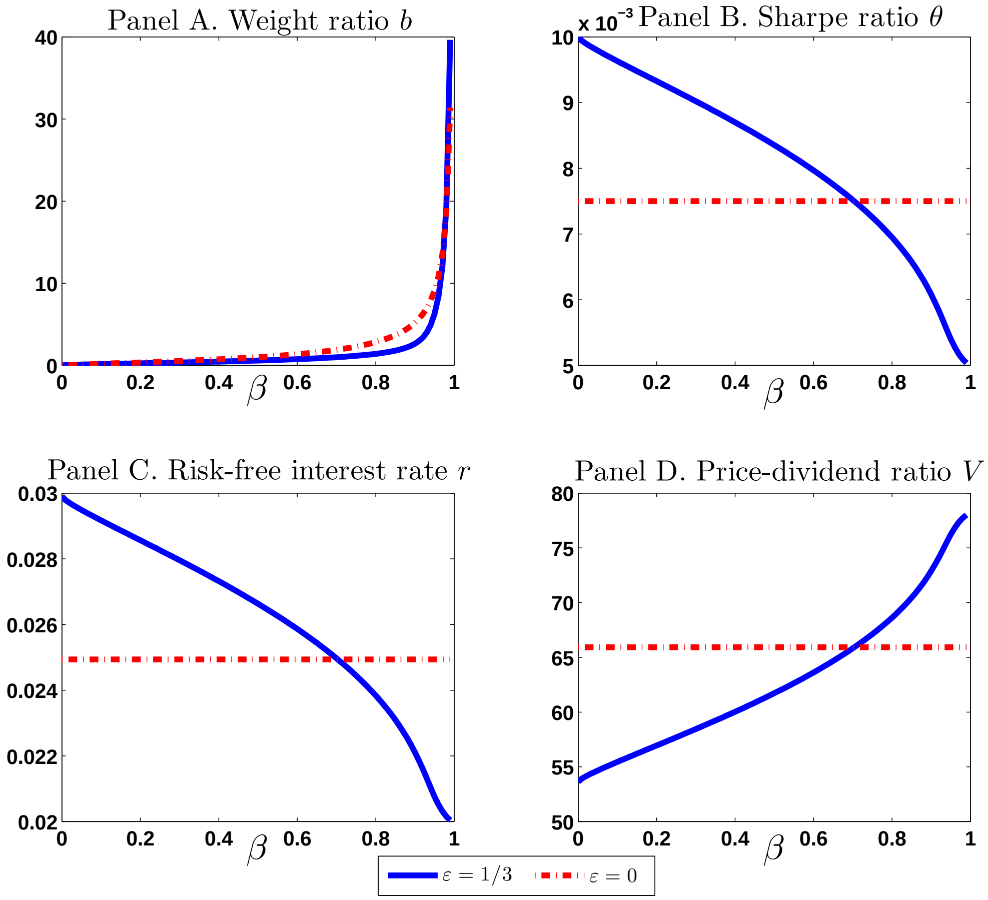

Figure 2: Effects of the size of institutions β. Panel A plots the ratio of the weight of institutional investors in the welfare function over the weight of retail investors in the welfare function b against the size of institutions β. Panel B plots the equilibrium Sharpe ratio θ against the faction of institutions in economy. Panels C and D plot the equilibrium asset prices: risk-free interest rate r and the price-dividend ratio V, respectively. The solid (blue) line corresponds to the equilibrium with heterogeneous investors; the dashed (red) line corresponds to an equilibrium in benchmark homogeneous economy. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.02, \sigma_D = 0.01, \varepsilon = 1/3, \gamma = 3/4, \delta = 0.01, T = 100, t = 20$.

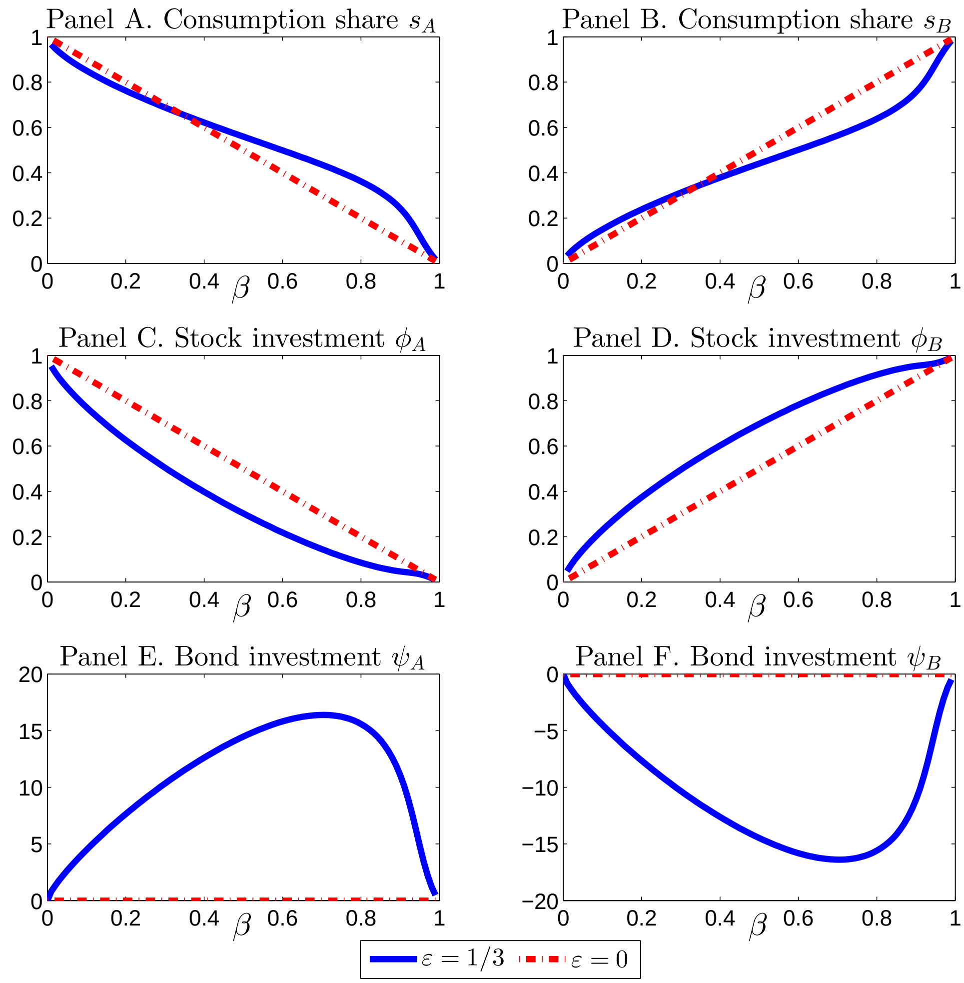

Figure 3: Effects of the size of institutions β. Panels A, C and E respectively plots the consumption share, holdings of the risk-free asset and holdings of the risky asset of retail investors. Panels B, D and F respectively plots that of institutional investors. The solid (blue) line corresponds to the equilibrium with heterogeneous investors; the dashed (red) line corresponds to an equilibrium in benchmark homogeneous economy. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.02, \sigma_D = 0.01, \varepsilon = 1/3, \gamma = 3/4, \delta = 0.01, T = 100, t = 20, D_t = 2.72$.

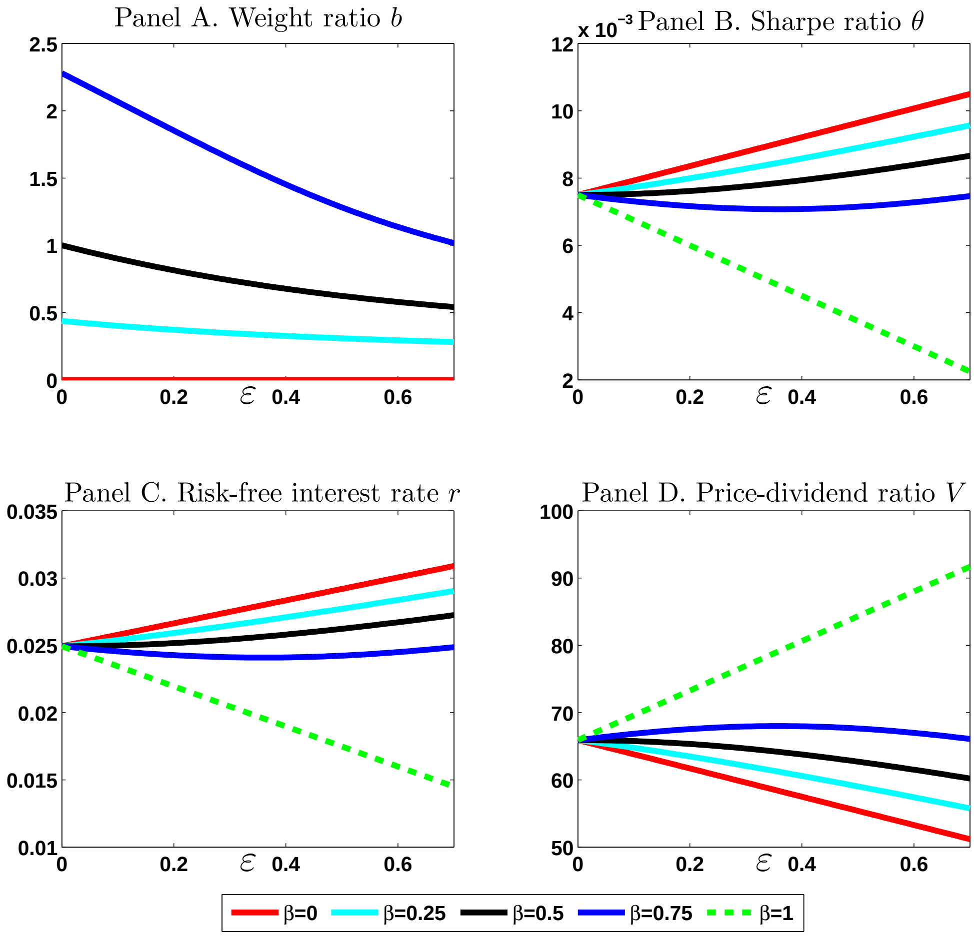

Figure 4: Effects of the risk-averse heterogeneity ε. Panel A plots the ratio of the weight of institutional investors in the welfare function over the weight of retail investors in the welfare function b against the size of institutions β. Panel B plots the equilibrium Sharpe ratio θ against the faction of institutions in economy. Panels C and D plot the equilibrium asset prices: risk-free interest rate r and the price-dividend ratio V, respectively. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.02, \sigma_D = 0.01, \gamma = 3/4, \delta = 0.01, T = 100, t = 20$.

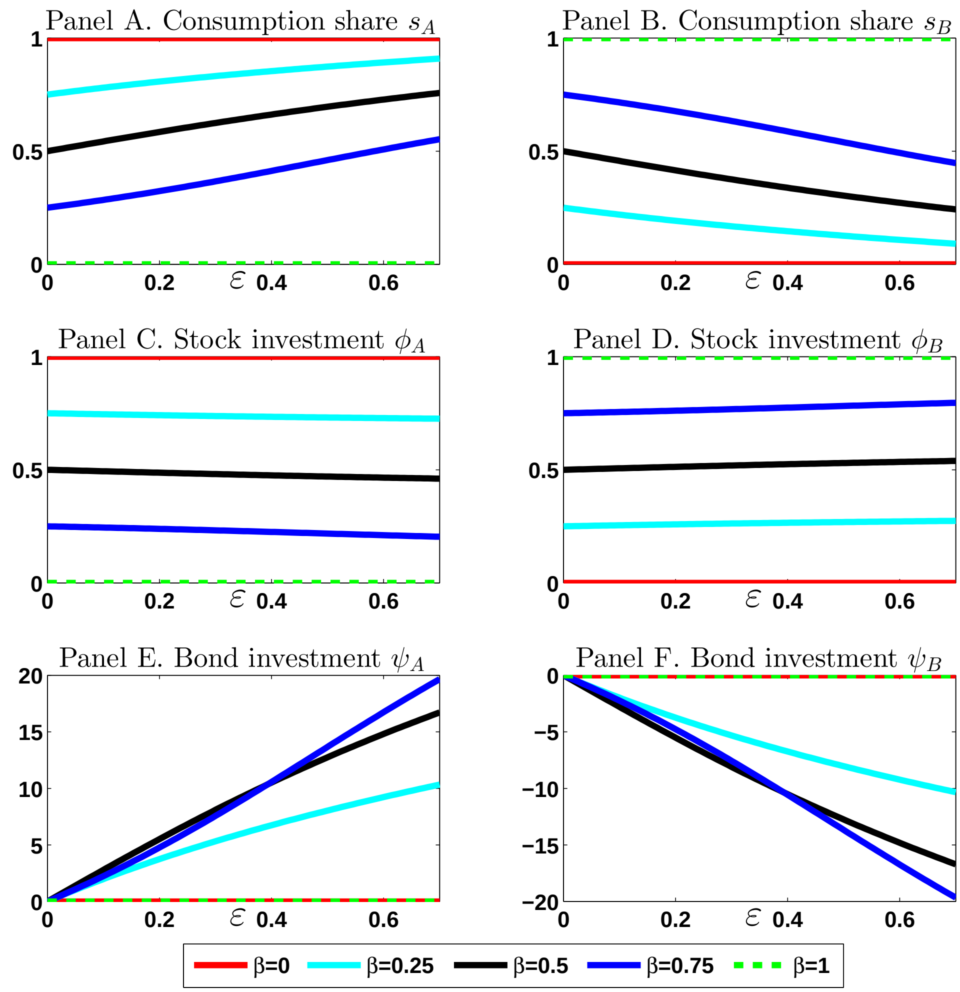

Figure 5: Effects of the risk-averse heterogeneity ε. Panels A, C and E respectively plots the consumption share, holdings of the risk-free asset and holdings of the risky asset of retail investors. Panels B, D and F respectively plots that of institutional investors. The plots are typical. The models parameters are as follows: $D_0 = 1, \mu = 0.02, \sigma_D = 0.01, \gamma = 3/4, \delta = 0.01, T = 100, t = 20, D_t = 2.72$.