Uncertainty and Investment Choice in a Real-Options Model of the Firm

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Abstract

We analyze the investment decision of a firm that has access to two distinct investment opportunities. An endogenous cash constraint means that the firm must not only choose the timing for each investment but must also often choose between investments. When compared to the unconstrained firm the constrained firm may either accelerate or delay investment in a given project depending on the level of cash it holds. If cash is sufficiently low then the firm is forced to delay investment until it has enough cash for investment. When projects are symmetric, the introduction of a second project causes the firm to delay investment as it waits to see which project is better. In some states the constrained firm will investment in a low-NPV, fast payback project over a higher NPV, slow payback project as the cash generated by the fast payback project better facilitates future investment. Investment in high pledgeability projects is also shown to be accelerated when the firm gains access to an additional project as these projects help to free up scarce resources for other investments.

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1 Introduction

There are many factors relevant to the investment decision, including, but not limited to, the size, scope and cost of each investment opportunity. It should come as no surprise then that the optimal investment decision is not a particularly easy one to make. McDonald and Siegel (1986) provide one of the first insights into the problem. They use a real options model to develop an optimal investment rule for an unconstrained firm with access to a single investment opportunity. Their work not only considers the payoff to investing but also the timing of investment and therefore the value associated with delaying investment. This model is later refined by Dixit and Pindyck (1994) in their book *Investment Under Uncertainty*; however, it is not until the work of Boyle and Guthrie (2003) that the problem of the cash constrained firm is explored. In their article they extend the model of McDonald and Siegel (1986) to include an endogenous financing constraint. The use of this constraint shows that due to the risk of future funding shortages, the constrained firm with a moderate level of cash will over invest and accelerate their investment when compared to the unconstrained firm. In contrast, constrained firms with sufficiently low liquidity will be forced to implement a delayed investment policy when compared to the unconstrained firm as insufficient resources are available for investment.

Following these initial studies there has been a recent surge in corporate finance literature that explores the interaction between constraints and investment. Cleary et al. (2007) uses empirical data to develop the idea of an investment curve that is u-shaped in cash. They show that, for low levels of internal funds, investment decreases monotonically, while for high levels of internal funds, investment is increasing monotonically. The same investment behaviour is described in Flor and Hirth (2013); however, they look at how the redeployability of assets affects investment decisions.[1] In their analysis it is shown that firms with highly liquid assets in place will experience a lower sensitivity of investment to internal funds. Shibata and Nishihara (2012) develop a model where the firm faces an exogenous debt capacity constraint. In their setup the firm can only raise a restricted amount of funds via debt issuance. Within their base case, a stronger debt financing constraint leads to a reduction in the credit spread and therefore a reduction in the probability of default. As such they find that the investment threshold is u-Shaped in the debt capacity constraint. Hirth and Uhrig-Homburg (2010a) show that due to stockholder-bondholder conflicts the constrained firm's investment threshold is monotonically decreasing in internal funds.

In addition to the aforementioned, costly external finance and its effect on investment has recently received a lot of exposure. Lyandres (2007) shows that investment-cash flow sensitivity is non-monotonic in the cost of external funds. For low (high) financing costs, investment-cash flow sensitivity is decreasing (increasing) in the level of cost. Hirth and Uhrig-Homburg (2010b) extend the model of Boyle and Guthrie (2003) to include issuance costs associated with raising external finance. Their results are qualitatively similar to those in Boyle and Guthrie (2003) for high and low liquidity firms; however, for moderate levels of liquidity, the investment threshold depends largely on the convexity of the cost function. Under a linear cost function the results are qualitatively similar to Bolton et al. (2014). However, Bolton et al. (2014) describes a large increase in the investment threshold when

[1] In their study redeployability measures the ease with which existing assets can be sold to other firms.

the firm is close to being able to cover the cost of investment entirely from internal funds. This occurs because the firm is willing to delay investment in order to avoid drawing on costly financing.

Although dynamic and informative in nature, a large proportion of the relevant literature revolves around a firm that is constrained not only in resources, but also in investment opportunities. That is to say, it explores models where the firm has access to a single investment project. In doing so it overlooks the interaction between investments, the dynamic effect of one investment on another and the associated investment choice problem.

Nishihara and Shibata (2013) provides a step in the right direction by investigating the investment policy surrounding expansions of differing scale. Another recent study looking at investment choice is that of Almeida et al. (2011), who use a three stage, high or low state model of investment. In the first stage the firm must allocate funds across three types of investments. Cash-generating investments generate cash in the period directly following the investment date. Collateralizable investments also generate cash in the period directly following investment; however, a proportion of their cash flow can be used as collateral for investment in the period prior to generation.[2] Risky investments are more valuable in expectation than non-risky investments; however, they produce less cash flow in the low state than their non-risky counterparts. In the second stage the firm has a new investment opportunity and must invest either a high or low amount. They find that firms distort their investment policy towards projects that generate collateralizable cash flow when faced with future financing constraints and prefer projects with faster payback periods.

Clementi and Hopenhayn (2006) also suggest that collateral is an important factor related to the investment dynamics of a firm. Through their use of a discrete-time, contracting model, they suggest that, in the extreme case, collateral constraints can cause investment opportunities to become financially infeasible. This suggests that when the collateral of a project is low, other investment opportunities can provide additional benefits over and above their NPV as the cash they generate helps to relax future collateral and financing constraints. This notion is empirically shown in Hennessy and Whited (2005), who investigate the costs associated with external financing. Gan (2007) also shows empirically that a reduction in asset collateral values leads to a reduction in investment. He uses a unique dataset of Japanese manufacturing firms that faced an exogenous shock to collateral values during the Japan asset price bubble of 1986-1991. The exogenous nature of this reduction in collateral solves the endogeneity problem generally associated with similar studies.

In addition to the aforementioned, the analysis completed by Thakor (1990) also suggests that a firm will prefer projects that have short payback periods. This preference is strongest when it is anticipated that future investments will need to be financed internally. Almeida and Campello (2007) also investigate the effect of asset pledgeability on investment. They find that the investment of constrained firms that have assets with low pledgeability will have a low sensitivity to cash flow.

In terms of investment choice and timing, Dixit (1993) presents a model where the firm must choose between projects of differing scale. The value of each project is a linear function

[2] It is worth noting that the literature uses both collateralizability and pledgeability interchangeably in reference to the amount of an asset's value that can be raised prior to investment by using the asset as collateral (pledging the asset).

of a single Geometric Brownian Motion and there is no follow-up investment. He suggests that as soon as the firm reaches the investment region for any single investment it will choose that project over the others. However, this conclusion is flawed as when investing in one project, the waiting value of some other project could be higher and therefore investment should be delayed. Décamps et al. (2006) address this problem and derive an optimal investment rule for a firm that must choose one investment out of a possible two. Investment scale is also of importance in Nishihara and Shibata (2013) who show that for a moderate level of cash, it is optimal to expand production on a smaller scale to avoid the costly financing of large scale expansions.

In order to explore the investment choice problem we extend the model of Boyle and Guthrie (2003) to include an additional investment opportunity. The resulting model and conclusions help to supplement the existing literature and further develop understanding of the constrained firm. It goes without saying that although the addition of an extra investment to the model in Boyle and Guthrie (2003) is a relatively simple extension, it provides numerous new insights. When projects are symmetric, the introduction of an extra project leads to a upwards shift of the original project's investment threshold. This investment delay occurs as the firm does not want to invest in one project to find out ex-ante it would have been better to do the other project. If projects are not symmetric, then the firm in some states will invest in a lower NPV, fast-payback project instead of a higher NPV, slow-payback project, as the project with a faster payback better facilitates follow-up investment after being launched via the cash it generates. If the firm instead has one project that can raise a relatively larger proportion of its value as debt for funding, then the introduction of an otherwise identical project with lower debt capacity will in some states shift the investment threshold downwards. In these states, investment in the high capacity project occurs earlier in order to free up cash that can be used for investment in the remaining project.

The model is set up in a progressive manner and solved backwards in project-time. First a cash constrained firm that has access to only one investment opportunity in a manner analogous to Boyle and Guthrie (2003) is explored. From here the model is built up to consider the same firm with one investment opportunity that now has access to existing assets in place which are resultant from previous investment decisions. The addition of these existing assets in place helps provide an insight into the optimal divestment policy of the firm whereby selling assets raises sufficient resources to fund investment. In its third stage the model is developed to show the investment decision of a firm with access to two distinct and separate investment projects. In this setting the firm not only decides on the optimal timing for each investment but in states where investment in either but not both projects is possible the firm must also choose *between* projects.

The remainder of this paper is organised as follows. Section 2 describes the investment environment and presents the investment choice model. Section 3 presents and discusses the constrained firms investment policy under both symmetric and two varieties of asymmetric projects. Lastly, Section 4 contains some concluding remarks.

2 Investment Environment

A natural extension to the model of Boyle and Guthrie (2003) is to investigate a firm that has access to more than one investment project. An unconstrained firm with perpetual rights to two projects will behave in a similar way to the one-project unconstrained firm and invest at the optimal levels for each project. The problem of the financially constrained firm now however becomes much more complex as it becomes not only a question of investment timing but also of investment choice. However, before exploring the case where the firm must choose between two investment opportunities, first the situation where the firm has to only choose the investment timing on one investment opportunity is explored. This is analogous to the work of Boyle and Guthrie (2003) although the frictions employed here provide a richer more realistic investment environment. Next, the situation where the firm has launched its initial investment and now has another investment opportunity is explored. The increased complexity of this problem stems from the previously undertaken investment opportunity. Because investment has already been undertaken in this first project, it is now generating cash which can help to relax the cash constraint of the firm. The set up is such that this new asset can be sold in order to raise capital to fund investment in the latest project. Selling existing assets comes at a cost to the firm which leads the firm to increase the threshold required for investment when assets in place must be sold to fund said investment. After this the final situation where the firm has rights to two investment projects and must choose the timing on both is explored in detail.

2.1 One Investment

Consider a firm that has perpetual rights to an investment project (project A) that, once initiated, generates a cash flow (A_t) at date t that evolves according to the following Geometric Brownian Motion

$$dA_t = \mu_A A_t dt + \sigma_A A_t d\xi_{A,t} \tag{1}$$

where μ_A and σ_A are constants that specify the drift and volatility of A respectively while $d\xi_{A,t}$ is the increment to a Weiner process. It follows that once launched the present value of project A at date t, $V_{A,t}$ will be equal to the current cash flow discounted at the risk adjusted discount rate less the cash flow growth rate. That is to say

$$V_{A,t} = \frac{A_t}{(r + \lambda_A) - \mu_A} \tag{2}$$

where r is the risk free interest rate and λ_A the risk premium of project A. Further simplification of Equation (2) can most readily be obtained by defining $r + \lambda_A \equiv \mu_A + \delta_A$, where δ_A represents the opportunity cost of cash flows forfeited due to waiting, more commonly known as the dividend yield. It follows that $V_{A,t} = A_t/\delta_A$, for further details see Dixit and Pindyck (1994).

In addition to the option rights, the firm has a cash stock X_t at time t and assets in place worth the constant G. Investment can occur at date t if and only if

$$I_A \leq X_t + G + \gamma_A \theta_A V_A \tag{3}$$

for some constants I_A, $\gamma_A \in [0,1)$ and $\theta_A \in [0,1)$. For the remainder of this section t subscripts will be dropped for simplicity as the explicit date at which investment occurs is not material. The left hand side of the investment constraint represents the lump sum cost that must be paid in order to initiate investment. The right hand side is composed of all the resources available to the firm to fund investment, cash plus the accessible value of the firm's assets. In this case the feasible value of the firm's assets represents the amount of short term risk free debt $(G + \gamma_A \theta_A V_A)$ that can be raised from creditors by pledging the assets in place G and the soon to be launched project A as collateral. The friction γ_A captures the information asymmetry between the firm and outside creditors and θ_A is a discount applied because the creditors do not have control of the project. It is in the use of these two frictions where the setup diverges from that of Boyle and Guthrie (2003) and will later be the driving factor in the firm divesting assets in place. Policy results under the two frictions are however qualitatively the same as their analysis. Without loss of generality no frictions are placed on G because there is no uncertainty around their value to creditors and therefore the firm may borrow up to their full value. The frictions on V_A mean that there will be states where the payoff to investment, $V_A - I_A$, is positive however the firm is unable to invest. These states occur when $X + G + \gamma_A \theta_A V_A < I_A < V_A$ and mean that the firm is forced to postpone investment.

As time goes on both X and G can help to relax the cash constraint. First, the cash balance is invested in risk free securities earning a rate of return r. Second, the existing assets G generate uncertain operational cash flow with dynamics $\nu dt + \phi d\zeta$, where ν and ϕ are constants and ζ is a Weiner process that satisfies $d\zeta d\xi_A = \rho_{XA} dt$ for some constant $\rho_{XA} \in [-1,1]$. The cash stock prior to investment therefore evolves according to the following process

$$dX = rXdt + \nu dt + \phi d\zeta. \tag{4}$$

Under Equation (4) the level of cash may go negative which represents the firm issuing short term debt to cover cash flow problems. Should the cash deficit exceed the accessible value of the firm's non-cash assets they will face liquidation[3] and be forced to sell the project rights for $F^u(\gamma_A A)$. In a competitive environment the buyer of the rights will be an unconstrained firm, accordingly $F^u(\gamma_A A)$ represents the value to the unconstrained firm evaluated at $\gamma_A A$. It is worth noting that when selling the rights the control friction θ_A is not applied because once sold the unconstrained buyer will be able to manage and control the project as they see fit. There still exists information asymmetry so γ_A is applied as usual. Shareholders are free to trade their shares on the open market so the objective of the firm will be to maximise its value, that is the sum of its cash, assets in place and the value of the investment option. Therefore maximising the value of the option will also maximise the value of the firm. The problem and solution of the unconstrained firm will first be provided as it serves as a useful benchmark as well as a limiting case for the constrained firm.

[3]Liquidation occurs when $X + G + F^u(\gamma_A A) \leq 0$ where $F^u(\gamma_A A)$ is the value to the unconstrained firm evaluated at $\gamma_A A$.

2.1.1 The Unconstrained Firm

The unconstrained firm has a sufficient cash balance such that in any state it can afford to invest. McDonald and Siegel (1986) were the first to investigate and provide a solution to this problem. Their efforts were later refined by Dixit and Pindyck (1994). Such a firm will invest if the current project cash flow exceeds some fixed investment threshold $\hat{A}^u$. Before investment the unconstrained option value, F^u, will satisfy the following ordinary differential equation

$$\frac{1}{2}\sigma_A^2 A^2 F_{AA}^u + (r - \delta_A) A F_A^u - r F^u = 0 \tag{5}$$

where subscripts on F^u denote partial derivatives.[4] In order to solve for the value of the option a few conditions are required. Firstly the option will be worth nothing when the project value is zero[5], $F^u(0) = 0$. Secondly at the investment threshold the option will be worth the payoff to exercising, $F^u(\hat{A}^u) = V_{\hat{A}^u} - I_A$. The second condition is the well known value matching condition that ensures the option value will be continuous at the investment boundary. The simplicity of the problem allows for an analytic solution of the option value before investment of the form

$$F^u(A) = (V_{\hat{A}^u} - I_A)\left(\frac{V_A}{V_{\hat{A}^u}}\right)^{\beta_A} \tag{6}$$

where

$$\beta_A = \frac{1}{2} - \frac{r - \delta_A}{\sigma_A^2} + \sqrt{\frac{2r}{\sigma_A^2} + \left(\frac{1}{2} - \frac{r - \delta_A}{\sigma_A^2}\right)^2} > 1. \tag{7}$$

Therefore given the above equations the option value to the firm will take on two forms, one when it is optimal to delay investment, $A < \hat{A}^u$, and another when investment is optimal, $A \geq \hat{A}^u$.

$$F^u(A) = \begin{cases} V_A - I_A & \text{if } A \geq \hat{A}^u \\ (V_{\hat{A}^u} - I_A)\left(\frac{V_A}{V_{\hat{A}^u}}\right)^{\beta_A} & \text{if } A < \hat{A}^u \end{cases} \tag{8}$$

Maximisation of Equation (8) provides the optimal investment threshold of

$$\hat{A}^u = \frac{\beta_A I_A \delta_A}{\beta_A - 1}. \tag{9}$$

It is worth noting that as $\beta_A > 1$ the investment threshold will be such that there will exist states where the payoff to investing, $V_A - I_A$, is positive although the firm chooses to delay.

2.1.2 The Constrained Firm

The problem faced by the constrained firm is that it does not have unlimited resources so there will exist states where it is unable to invest even if it would like to. The value of the

[4] Details of the derivation can be in Dixit and Pindyck (1994)

[5] This is an absorbing barrier in that upon reaching zero a Geometric Brownian Motion will stay there indefinitely, so that the option will never be exercised and is therefore worthless.

Table 1: This table outlines the parameter values used to numerically solve Equation (11) for the optimal investment timing policy. Most values are similar to those used in Boyle and Guthrie (2003). The frictions on investment however are instead represented by two separate parameters: information asymmetry as represented by γ_A and the control discount represented by θ_A.

Project investment cost ($)	$I_A = 100$
Project cash flow volatility	$\sigma_A = 0.20$
Project dividend yield	$\delta_A = 0.03$
Information asymmetry	$\gamma_A = 0.80$
Control discount	$\theta_A = 0.40$
Riskless interest rate	$r = 0.03$
Firm cash flow - Project A cash flow correlation	$\rho_{XA} = 0.50$
Firm cash flow volatility	$\phi = 60$
Market value of existing assets	$G = 100$

project rights now becomes a function of both cash and project cash flow denoted $F(X, A)$. Like the unconstrained problem there will be an investment threshold, $\hat{A}$, whereby if $A \geq \hat{A}$ the firm will invest and receive a payoff worth $V_A - I_A$. However, now $\hat{A}$ is a function of X rather than a constant as in the unconstrained case. In a similar fashion to the unconstrained firm the option will be worthless when the project cash flows are zero, $F(X, 0) = 0$ and the option will be worth the exercise payoff at the investment boundary, $F(X, \hat{A}) = V_{\hat{A}} - I_A$. As the cash balance increases the constraint is relaxed and the value of the option will approach that of the unconstrained case,

$$\lim_{X \to \infty} F(X, A) = F^u(A). \tag{10}$$

When A is below the investment threshold then the firm will delay investment and the option value will satisfy the partial differential equation

$$\frac{1}{2}\sigma_A^2 A^2 F_{AA} + \rho_{XA}\sigma_A \phi A F_{XA} + \frac{1}{2}\phi^2 F_{XX} + (r - \delta_A) A F_A + r(X + G) F_X - rF = 0 \tag{11}$$

where subscripts on F denote partial derivatives.[6] The complexity of this equation means than an analytical solution is not available and therefore numerical methods are required to solve it.

Before discussion of the solution it is worthwhile to identify the situations the firm may find itself in for any state. There exists a total of three scenarios that will dictate the actions available to the firm. Firstly, if $X + G + F^u(\gamma_A A) \leq 0$ the firm is within the liquidation region and it is forced to sell the rights for $F^u(\gamma_A A)$ as above. This corresponds to Region 1 of Figure 1 that has been constructed using the values from Table 1. The figure demonstrates that as the potential project cash flows and therefore the present value of a launched project increase, the firm is able to avoid liquidation for more and more negative levels of cash.

[6]Details of the derivation can be found in the Appendix.

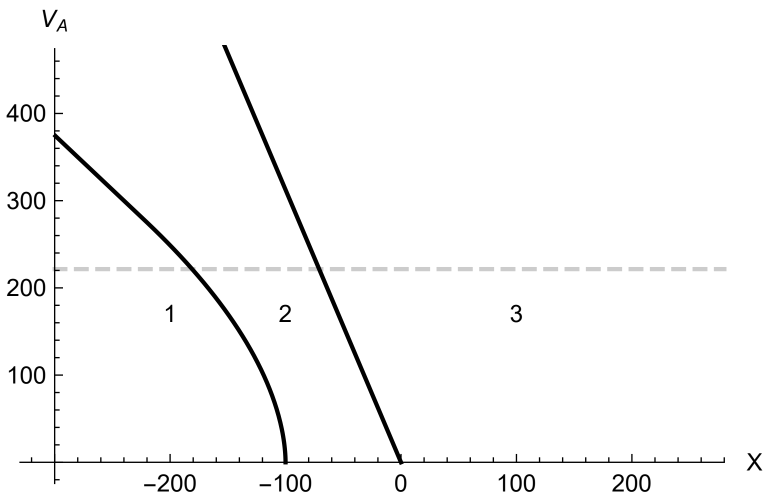

Figure 1: This figure shows the three regions the constrained firm can be in using the parameters of Table 1. The horizontal dashed line is the unconstrained investment threshold. Region 1 corresponds to the liquidation region where $X + G + F^u(\gamma_A A) \leq 0$. It is here where the firm is forced to sell the rights to the project for $F^u(\gamma_A A)$. Region 2 is the forced waiting region. Here the firm has sufficient resources to continue operation, $X+G+\gamma_A\theta_A V_A > 0$ but does not have enough resources to fund investment, $I_A > X+G+\gamma_A\theta_A V_A$. Within region 3 the firm is able to invest as they have sufficient resources to do so, $I_A \leq X + G + \gamma_A\theta_A V_A$.

Secondly, the firm can be in a state where it has enough resources to continue operation even though it does not have enough to undertake investment. This will occur when both $X + G + F^u(\gamma_A A) > 0$ (the firm is liquid) and $I_A > X + G + \gamma_A\theta_A V_A$ (the firm is unable to invest). These states where forced delay occur correspond to Region 2 of Figure 1. Even if the firm would like to invest here they are unable to, although they may be able to invest in the future if the constraints are relaxed. Finally, the firm will be able to choose between investing or waiting when $I_A \leq X + G + \alpha_A\gamma_A V_A$ as shown by Region 3 of Figure 1. Similar to the liquidation region the firm has the choice to invest for more and more negative values of cash as the project cash flows increase. It is worth noting that no matter how large V_A is there will always exist a level of cash that forces the firm to delay investment. This result is in contrast to Boyle and Guthrie (2003) who only employ a single investment friction. The use of a single investment friction means that there will be a critical level of project cash flows that once surpassed the firm will find itself either liquidated or able to invest for any level of cash. The grey dashed line of Figure 1 shows the investment threshold of the unconstrained firm.

2.1.3 A Numerical Solution

In order to provide a solution to Equation (11) numerical methods based on finite differences and successive over relaxation are implemented, the details of which can be found in the Appendix. The impact of being constrained is best shown by the investment policy undertaken by the firm. Refer to Figure 2, which plots the regions and unconstrained threshold of Figure 1 as well as the optimal investment threshold $\hat{A}$ (the black curve). First consider the unconstrained firm that will invest when the value of the project equals or exceeds \$221.53. In comparison to this, when the constrained firm has considerably lower levels of cash, the

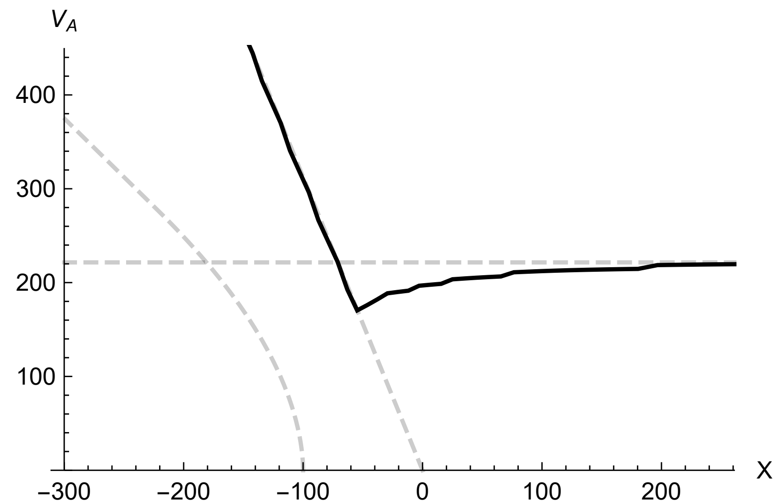

Figure 2: This figure shows the investment threshold of the constrained firm under the parameters in Table 1. The grey dashed lines correspond to the unconstrained threshold, liquidation constraint and investment constraint of Figure 1. The black curve is the constrained firm investment threshold.

financing constraint binds and $\hat{A} > \hat{A}^u$. As the constraint binds the firm invests as soon as possible due to the risk of future funding shortfalls being such that the firm can not afford to miss out. The more negative the cash balance the further the constrained firm delays investment when compared to the unconstrained firm. For moderately low levels of cash the investment constraint still binds however the firm accelerates investment compared to that of the unconstrained firm, that is they over invest. In this region the threat of future funding shortfalls is still great enough that the constrained firm chooses to invest as soon as possible in order to avoid losing the investment opportunity in the future. As the cash balance increases, the constraint no longer binds although the firm still invests earlier than the unconstrained level. The level of acceleration decreases with cash as the risk of funding shortfalls gets smaller and smaller. Eventually the firm has so much cash that, due to the extremely low shortfall risk, the firm invests at the unconstrained level.

2.2 Two Investments: One Launched, One Opportunity

Now imagine that the investment opportunity in Section 2.1 has been undertaken. The firm's assets in place now consist of G plus project A, which generates cash according to Equation (1). Consider that a new investment is now available to which the firm has perpetual rights. Let this opportunity be called project B that once initiated generates cash flow that evolves according to the following Geometric Brownian Motion

$$dB = \mu_B B dt + \sigma_B B d\xi_{B,t} \tag{12}$$

where μ_B and σ_B are constant parameters that specify the drift and volatility of the cash flow respectively and $\xi_{B,t}$ is a Weiner process that satisfies $d\xi_A d\xi_{B,t} = \rho_{AB}$.[7] It follows from

[7]If the firm did not have project A launched and in place, then the option value of project B can be calculated in the same manner as the previous section and is represented by $F^B(X, B)$.

earlier reasoning that once launched the project will have a present value (V_B) equal to

$$V_B = \frac{B}{(r+\lambda_B)-\mu_B} = \frac{B}{\delta_B} \tag{13}$$

where λ_B is the risk premium of B and δ_B the implicit dividend yield.

As time goes on X, G and A can help to relax the cash constraint. Similar to before the cash balance is invested in risk free securities earning a rate of return r and G generates uncertain operational cash flow with dynamics $\nu dt + \phi d\zeta$ where ζ is a Weiner process that satisfies $d\zeta d\xi_{B,t} = \rho_{XA}$ and $d\zeta d\xi_{B,t} = \rho_{XB}$. In addition to this, project A will generate cash according to Equation (1) resulting in an overall evolution of the cash stock prior to investment of

$$dX = rXdt + \nu dt + \phi d\zeta + Adt. \tag{14}$$

Analogous to the single project case, the cash stock here is able to go negative and depending on its magnitude, the firm may be forced into liquidation for sufficiently negative values. In this case forced liquidation will occur if $X + G + \gamma_A V_A + F^{UB}(\gamma_B B) \leq 0$. Once cash is low enough so as to trigger this forced liquidation, the investment rights are sold for $F^{UB}(\gamma_B B)$ which is the value of the rights to an unconstrained firm evaluated at $\gamma_B B$.

Unlike the single project set up of Section 2.1 the firm now has existing assets of G as well as project A. The original existing assets G carry no frictions so either selling or borrowing against them will provide the same level of resources. In contrast to the original assets, the borrowing capacity of project A differs from its selling capacity. Investment in this second project without divesting existing assets can occur if and only if

$$I_B \leq X + G + \gamma_A \theta_A V_A + \gamma_B \theta_B V_B. \tag{15}$$

The left hand side of Equation (15) represents the lump sum investment cost of investing while the right hand side represents all the resources available to fund investment. That is, cash plus the debt capacity of assets in place plus the borrowing capacity project B. Both the γ_A and θ_A frictions are still applied as there exists information asymmetry as well as a control discount for project A. Once again without loss of generality there are no frictions placed on G. The borrowing capacity of the new project is represented by $\gamma_B \theta_B V_B$ where the constants $\gamma_B \in [0, 1)$ and $\theta_B \in [0, 1)$ are borrowing frictions applied due to information asymmetry and creditors not having control over the project respectively.

The firm is also able to divest a proportion of project A in order to fund investment. Consider the case where the firm does not utilise the borrowing capacity of project A, investment requires that $I_B \leq X + G + \gamma_B \theta_B V_B$. If this does not hold then the firm needs to raise

$$J_A = I_B - X - G - \gamma_B \theta_B V_B \tag{16}$$

from Project A to fund investment. As detailed in Equation (15) riskfree debt of up to $\gamma_A \theta_A V_A$ can be undertaken. This represents the maximum capacity of how much the firm can raise from the external capital market via creditors provided that project A is pledged as collateral. The firm may also sell a proportion of the asset to an outside investor where selling the full asset provides capital of $\gamma_A V_A$. Selling does not incur the θ_A friction because once launched the project is assumed to be entirely divisible. As such the borrowing capacity of project A is less than what can be raised from selling. Therefore in order to invest by divesting some proportion of project A it must be such that

$$I_B \leq X + G + \gamma_A V_A + \gamma_B \theta_B V_B. \tag{17}$$

The two financing options are not mutually exclusive so a combination of borrowing and selling can be implemented. If a proportion $\psi_{BA} \in (0, 1]$ is borrowed against and $\psi_{SA} \in (0, 1]$ sold then

$$J_A \leq \psi_{BA} \gamma_A \theta_A V_A + \psi_{SA} \gamma_A V_A \tag{18}$$

in order to invest.[8] The optimal mix of borrowing and selling is motivated by the idea that where possible the firm will prefer to borrow instead of selling. This notion is in line with the pecking order theory as discussed in Myers and Majluf (1984). It also mirrors the results presented in Lang et al. (1995) where a firm will sell assets only when alternative funding is insufficient. In contrast to borrowing, selling is costly to the firm. When divesting a proportion ψ_{SA} of project A the firm will give up $\psi_{SA} V_A$ of asset value but would have only received $\gamma_A \psi_{SA} V_A$ for it, as such the overall payoff is

$$\psi_{SA} V_A (\gamma_A - 1) < 0. \tag{19}$$

The loss made on divesting illustrates why the firm prefers to borrow where possible. It is important to note that any policy undertaken must be within the firm's feasible set. That is to say it is not possible to utilise the full debt capacity of the project whilst also selling the rights to the entire project, $\psi_{BA} + \psi_{SA} \leq 1$. Partial divestment is possible in a sense that the firm can sell an ownership stake within the existing asset. The overall payoff when divesting to raise capital and subsequently investing will be the investment payoff $V_B - I_B$ less the loss on divestment. In order to maximise firm value the firm will choose ψ_{BA} in order to minimise their loss, as such their objective function is

$$\begin{aligned} \min \quad & \psi_{SA} \\ \text{s.t.} \quad & J_B = \psi_{BA} \gamma_A \theta_A V_A + \psi_{SA} \gamma_A V_A \\ & \psi_{BA} + \psi_{SA} \leq 1 \\ & \psi_{SA} \geq 0 \end{aligned} \tag{20}$$

The problem can best be illustrated via a graph of the constraints as seen in Figure 3. On the left is a graph that demonstrates the case where J_B can be raised entirely by borrowing a proportion equal to ψ^*_{BA}. The black line represents all combinations of ψ_{BA} and ψ_{SA} that allow the firm to raise J_B while the grey line illustrates the limit of resources that can be raised from project A, $\psi_{BA} + \psi_{SA} = 1$. In the left hand case the payoff to investment will be the same as in the standard case, $V_B - I_B$. As J_B increases the dark black line will shift outwards until the borrowing capacity is not sufficient for investment. The graph on the right shows a situation where this has occurred and partial divestment is required to fund investment. In this instance the J_B line intersects the funding constraint and therefore both borrowing and divestment is required with corresponding levels of ψ^*_{BA} and ψ^*_{SA}. Solving

[8] The constraint will be binding as the setup is such that there is no incentive for the firm to raise and

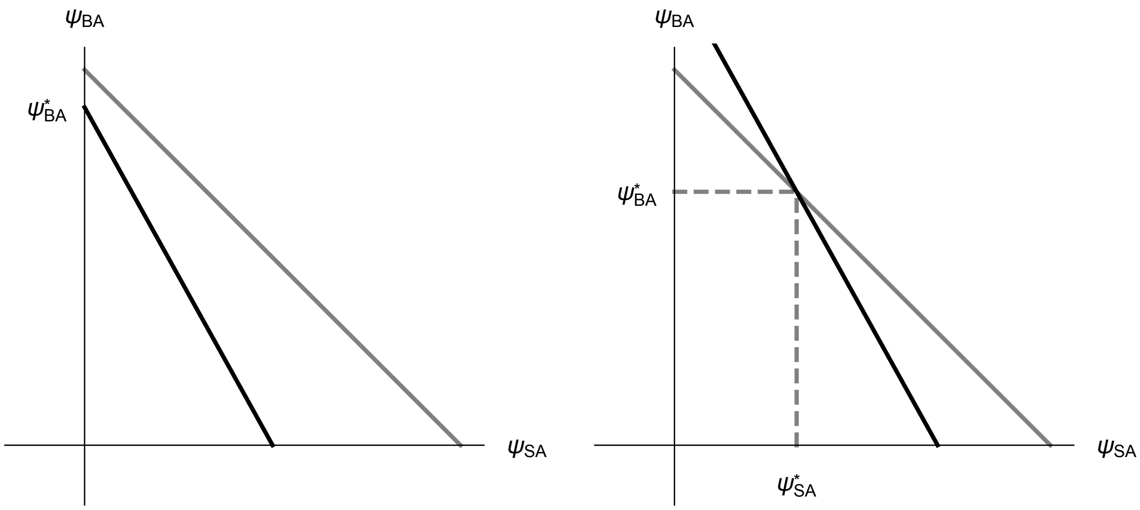

Figure 3: These graphs demonstrate the general behaviour of the optimal divestment policy. The black line on each graph represents the amount of extra funding required to be raised from project A in order to invest in project B. The grey line illustrates the funding constraint where $\psi_{BA} + \psi_{SA} = 1$. The graph on the left hand side shows a situation where the firm can raise the extra funds required to invest in project B by solely using a percentage ψ^*_{BA} of the borrowing capacity of project B. The right hand graph shows a situation where the firm must use borrowing capacity ψ^*_{BA} of project A as well as as divest a proportion ψ^*_{SA} of project A in order to fund investment in project B.

Equation (20) provides an analytic solution for ψ_{SA} of the form

$$\psi_{SA} = \frac{J_B}{\gamma_A(1-\theta_A)V_A} - \frac{\theta_A}{1-\theta_A}. \tag{21}$$

Upon substitution and rearrangement the payoff to investing in project B via divestment in project A will therefore be equal to

$$(V_B - I_B) - \left(\frac{J_B}{\gamma_A} - \theta_A V_A\right)\left(\frac{1-\gamma_A}{1-\theta_A}\right). \tag{22}$$

It is worth remembering that this payoff will only be applicable when the maximum borrowing capacity of project A does not raise sufficient capital for investment. Before further discussing the constrained firm, the unconstrained case is illustrated below.

2.2.1 The Unconstrained Firm

The situation of the unconstrained firm will not change as they can still invest when they want without ever needing to sell existing assets. The option value to such a firm will be

$$F^{UB}(B) = \begin{cases} V_B - I_B & \text{if } B \geq \hat{B}^u \\ (V_{\hat{B}^u} - I_B)\left(\frac{V_B}{V_{\hat{B}^u}}\right)^{\beta_B} & \text{if } B < \hat{B}^u \end{cases} \tag{23}$$

horde extra cash.

where

$$\beta_B = \frac{1}{2} - \frac{r-\delta_B}{\sigma_B^2} + \sqrt{\frac{2r}{\sigma_B^2} + \left(\frac{1}{2} - \frac{r-\delta_B}{\sigma_B^2}\right)^2} > 1. \tag{24}$$

The optimal investment threshold will be

$$\hat{B}^u = \frac{\beta_B I_B \delta_B}{\beta_B - 1}. \tag{25}$$

2.2.2 The Constrained Firm

The objective of the constrained firm is to choose an investment threshold that maximises its option value. This value now becomes a function not only of cash and potential cash flows but also of the cash flow being generated by Project A, denoted $F^{BA}(X,A,B)$ where F^{BA} refers to the option to invest in project B given that project A has already been initiated. Similar to the one project case, the option will be worthless when the project cash flows are zero, $F^{BA}(X,A,0) = 0$. When the cash flows of Project A are zero then the solution simplifies[9] down to the single project case of section 2.1, $F^{BA}(X,0,B) = F^B(X,B)$. As the cash balance increases the financing constraint is relaxed and the value of the option will approach that of the unconstrained firm,

$$\lim_{X\to\infty} F^{BA}(X,A,B) = F^{UB}(B). \tag{26}$$

As the cash flows generated by project A increase and the financing constraint is relaxed the value of the option will approach that of the unconstrained firm,

$$\lim_{A\to\infty} F^{BA}(X,A,B) = F^{UB}(B). \tag{27}$$

Prior to investment the option value will satisfy the following partial differential equation

$$\frac{1}{2}\sigma_A^2 A^2 F_{AA}^{BA} + \frac{1}{2}\sigma_B^2 B^2 F_{BB}^{BA} + (r-\delta_A)AF_A^{BA} + (r-\delta_B)BF_B^{BA} + (r(X+G)+A)F_X^{BA} + \frac{1}{2}\phi^2 F_{XX}^{BA} + AB\rho_{AB}\sigma_A\sigma_B F_{AB}^{BA} + \rho_{XB}\sigma_B B\phi F_{XB}^{BA} + \rho_{XA}\sigma_A A\phi F_{XA}^{BA} - rF^{BA} = 0 \tag{28}$$

where subscripts on F^{BA} denote partial derivatives.[10] Again the complexity of this equation precludes an analytic solution so numerical techniques are used to find a solution. Before discussion of the solution, it is worthwhile to assess the states the firm may find itself in and how they are affected. With the added complexity of assets in place the firm may find itself in one of four situations depending on the level of X, A and B. Refer to Figure 4, which has been constructed using the parameter values of Table 2. Each graph is analogous to Figure 1, but drawn for a different level of "other-project" cash flows. The top left graph shows the situation where the cash flows of Project A are equal to \$0. Due to the characteristics of the cash flow process, \$0 is an absorbing barrier and therefore the project will generate no further

[9] When $A = 0$, project A is so bad that it is generating no cash.

[10] A full derivation of Equation (28) can be found in the Appendix.

Table 2: This table outlines the parameter values used to numerically solve Equation (28) for the optimal investment timing policy. The parameters of each project are identical and where possible similar to those of Boyle and Guthrie (2003).

	Project A	Project B
Project investment cost ($)	$I_A = 100$	$I_B = 100$
Project cash flow volatility	$\sigma_A = 0.20$	$\sigma_B = 0.20$
Project dividend yield	$\delta_A = 0.03$	$\delta_B = 0.03$
Information Asymmetry	$\gamma_A = 0.80$	$\alpha_B = 0.80$
Control friction	$\theta_A = 0.60$	$\theta_A = 0.60$
Riskless interest rate	$r = 0.03$	
Firm cash flow-Project A cash flow correlation	$\rho_{XA} = 0.5$	
Firm cash flow-Project B cash flow correlation	$\rho_{XB} = 0.5$	
Project A cash flow-Project B cash flow correlation	$\rho_{AB} = 0.5$	
Firm cash flow volatility	$\phi = 60$	
Market value of existing assets	$G = 100$	

cash flow. In such a scenario the investment problem becomes equivalent to that of Section 2.1. In Region 1, $X+G+\gamma_A V_A+F^u(\gamma_B B) \leq 0$ so the firm is liquidated and the project rights are sold for $F^u(\gamma_B B)$. In Region 2 the firm is liquid, $X+G+\gamma_A V_A+F^u(\gamma_B B) > 0$, however is unable to invest, $I_B > X + G + \theta_B \gamma_B V_B$ so they are forced to postpone investment. Region 3 is not present as Project A is worthless and therefore can not be used to raise resources for investment. In Region 4 the firm has sufficient resources to invest if it wishes to, $I_B \leq X + G + \gamma_B \theta_B V_B$.

The top right graph illustrates the applicable regions where the present value of Project A is equal to $67. Regions 1 and 2 still represent liquidation and forced waiting respectively. Region 3 is applicable when the firm is able to invest although doing so requires divestment in Project A, that is $\gamma_A \theta_A V_A < J_B \leq \gamma_A V_A$. Along the line separating Regions 2 and 3 is where the firm must sell Project A in its entirety to fund investment. Region 4 corresponds to the situation where the firm is able to invest without divesting, $I_B \leq X+G+\gamma_A\theta_A V_A+\gamma_B\theta_B V_B$. Moving from the top right graph to the bottom left followed by the bottom right the present value of Project A increases from $67 to $141 and finally to $200. Multiple flow-on effects occur within the choices available to the firm. Firstly, as project A generates more cash the firm is able to sustain operation for more and more negative cash balances. Secondly, the firm is able to invest via divestment for more and more negative levels of cash. Not only this but the applicable divestment region increases in width as the funds that can be raised increase with V_B. Finally, the firm is also able to invest without divesting for more and more negative cash balances. The next section will discuss an intuitive solution to the investment threshold within these regions.

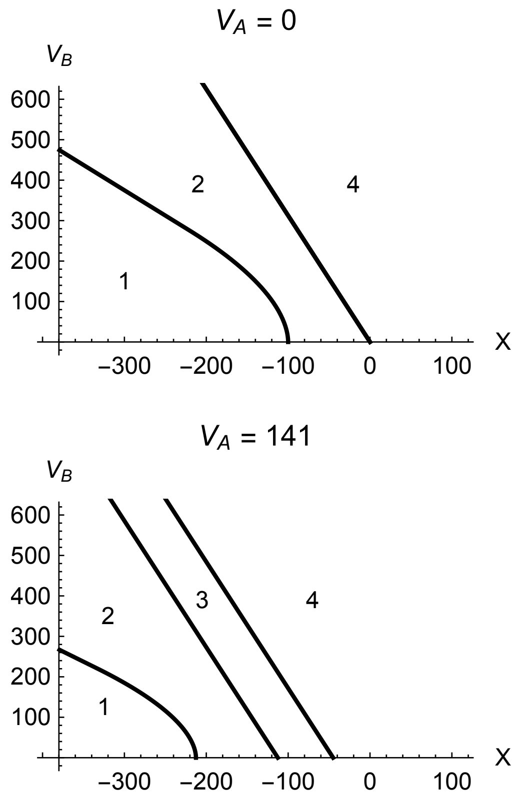

Figure 4: This figure shows the regions the constrained firm, with one project launched and one yet to be launched, under the parameters of Table 2. Because one project has already been initiated, the regions will depend on how much cash this launched project is generating. The displayed graphs have values of V_A equal to \$0, \$67, \$141 and \$200. In Region 1 the firm is forced to liquidate. In Region 2 the firm is forced to delay investment. In Region 3 the firm can invest but must divest a proportion of Project A in order to do so. In Region 4 the firm is able to invest without divesting any assets in place.

2.2.3 An Intuitive Solution

The frictions associated with investment will in some states force the constrained firm to delay although they would like to invest if they were unconstrained. In addition the constrained firm will in some states choose to divest a proportion of existing assets in order to raise capital for investment, a costly exercise. This behaviour implies that

$$F^{BA}(X, A, B) \leq F^{u}(B). \tag{29}$$

It is the constraints placed on the firm that force it into utilising a suboptimal investment policy. Depending on the levels of the cash stock (X) and project value (V_A) the constrained firm may either postpone (under invest) or accelerate (over invest) investment when compared to the unconstrained firm. Delayed investment can be either voluntary or involuntary. Involuntary delay occurs when the financial constraints are severe enough to prevent investment regardless of whether the firm would like to invest or not. Voluntary delay happens when although the firm is able to invest to do so would require the costly divestment of assets in place. If the project value is not sufficiently large to justify the divestment cost the constrained firm chooses to delay. Acceleration of investment happens when the firm has

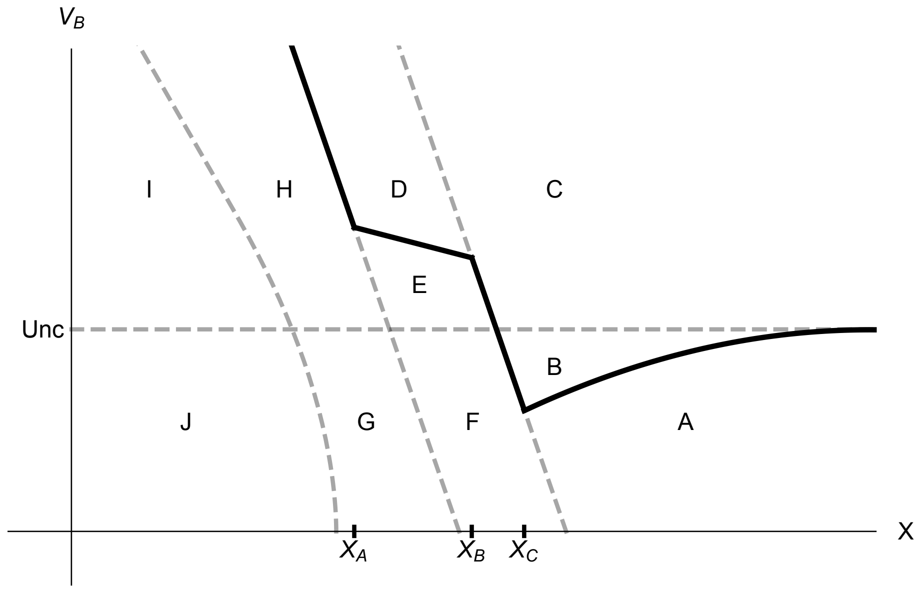

Figure 5: This figure shows the general shape of the constrained optimal investment threshold for some $A > 0$. The constrained firm will choose to invest for values of V_B and X that are above the black curve. The grey dashed lines illustrate the investment, investment via divestment and liquidation constraints as well as the unconstrained threshold in a similar fashion to Figure 4. In region A neither the constrained nor the unconstrained firm chooses to invest. In region B the constrained firm chooses to invest due to the risk of not having the resources to do so in the future while the constrained firm is able to wait. In region C, V_B is sufficiently large that both the constrained and unconstrained firms will invest. In region D, V_B is large enough that the constrained firm is willing to invest by selling a proportion of assets in place in order to raise capital, the unconstrained firm also invests. In region E in order to invest the constrained firm would need to divest assets which is costly, V_B is not large enough to justify this cost so investment is postponed while the unconstrained firm invests. In region F neither the constrained nor the unconstrained firm choose to invest. In regions G and H the constrained firm is forced to postpone investment while the unconstrained firm postpones and invests respectively. In regions I and J the constrained firm is liquidated while the unconstrained firm invests and postpones respectively.

the resources required for investment but the threat of future shortages leads them to invest early.

Figure 5 displays the intuitive constrained investment threshold for an arbitrary $A > 0$. The result is qualitatively similar to the one described in Hirth and Uhrig-Homburg (2010b) under linear issuance costs. For a low X, $X < X_A$, the investment threshold is high and steeply increasing as X gets lower and lower. This behaviour occurs because the investment constraint where the entire capacity of divestment occurs is binding. In order to avoid the risk of future shortfalls, investment occurs as soon as possible regardless of the cost associated with divesting as the firm does not want to miss out on the project. For moderately low levels of cash, $X_A < X < X_B$, the firm chooses to invest although doing so requires the divestment of assets in place. Within this range the threshold decreases in cash because increasing cash relaxes the cash constraint and means the firm can divest a smaller proportion and still afford investment. As cash increases the friction-driven cost of investment decreases and the firm is willing to invest for a lower project value. For a moderately high level of cash, $X_B < X < X_C$, the investment without divestment constraint is binding. For lower levels of X within this range there is underinvestment while higher levels lead to over investment.

The firm is averse to selling because doing so is costly, which means they are willing to either delay until a time where V_B is relatively large and justifies the divestment cost or accelerate investment in order to avoid having to sell to invest in the future. For high levels of cash, $X > X_C$, the firm accelerates investment in order to avoid the risk of future financing constraints. As cash gets larger and larger the level of acceleration reduces and the threshold approaches that of the unconstrained firm.

Figure 5 also demonstrates the action taken by the constrained firm for a given state. In region A both the constrained and unconstrained firms choose to delay investment. In region B the constrained firm invests in order to avoid the risk of future resource shortfalls while the unconstrained firm delays. Within region C both the constrained and unconstrained firms invest. In region D, V_B is large enough that the constrained firm is willing to divest some assets at a cost in order to fund investment; the unconstrained firm also invests. In region E, V_B is not large enough to justify the cost associated with divesting assets in order to raise capital so the constrained firm delays investment while the unconstrained firm invests. Both the constrained and unconstrained firms will choose to delay investment when found to be in region F. In regions G and H the constrained firm is forced to delay investment because it lacks the resources to invest. With its increased flexibility the unconstrained firm chooses to delay in region G and invest in region H. When finding itself in region I and J the constrained firm is still unable to invest although now its lack of resources forces it into liquidation where it must sell the option rights. In contrast the unconstrained firm is able to continue operation and invests when in region I and delays when in region J.

2.2.4 A Numerical Solution

The above intuition is confirmed with the use of numerical methods based on finite difference equations and successive over relaxation to solve Equation (28) followed by further intuitive analysis. The parameters shown in Table 2 are used for the baseline case. The results of the numerical solution procedure for a variety of V_B are displayed in Figure 6 which superimposes the optimal investment threshold on Figure 4. The top left graph corresponds to the case where $A = 0$ and therefore project A is worthless, thus the investment policy is comparable to the one in Section 2.1, but with project B the subject rather than project A. The remaining graphs show the policy for levels of V_A equal to \$67, \$141 and \$200. Recall from Figure 4 that as the cash flows of Project A increase, the firm is able to continue operation for more and more negative values of cash. Accordingly the investment threshold shifts leftwards with the investment and liquidity constraints.

The numerical solution backs up the intuition behind the intuitive solution discussed prior.

2.3 Two Investments: None Launched, Two Opportunities

Finnaly, consider the case where the firm has perpetual rights to two projects, Project A and Project B, with no existing assets other than G. Once initiated the investment opportunities will generate cash flow that evolve according to the following Geometric Brownian Motions,

$$dA = \mu_A A dt + \sigma_A A d\xi_A \tag{30}$$

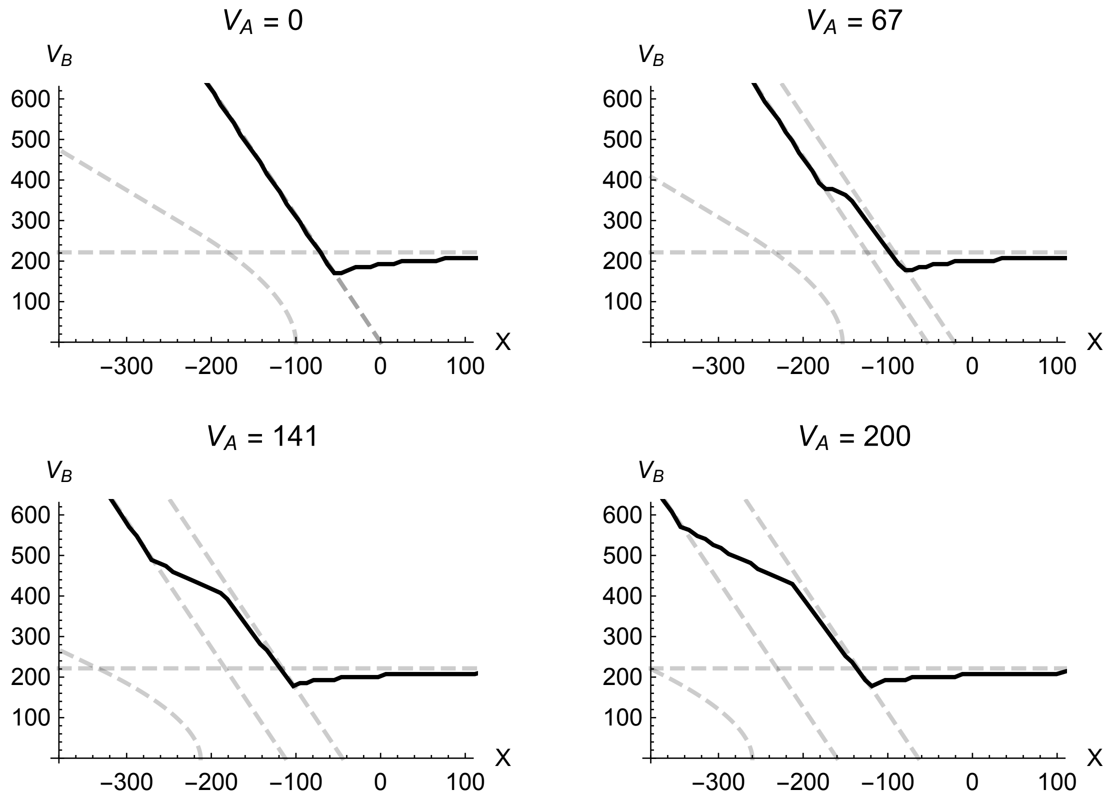

Figure 6: This figure shows the numerical solution to the constrained investment threshold under the parameters in Table 2. The top left graph shows the investment threshold when the value of the launched project is 0 and therefore the solution shows the same characteristics as the one in Section 2.1. The remaining graphs show the optimal investment policy when V_A is equal to \$67, \$141 and \$200.

and

$$dB = \mu_B B dt + \sigma_B B d\xi_B \tag{31}$$

respectively where μ_A, μ_B are constants that specify each project's drift, σ_A, σ_B are constants that specify each project's volatility and ξ_A, ξ_B are the increments to Weiner processes that satisfy $d\xi_A d\xi_B = \rho_{AB} dt$. As detailed previously, Project A and Project B will have present values of A/δ_A and B/δ_B respectively once completed.

The cash balance of the firm evolves in a manner identical to that of Section (2.1),

$$dX = rXdt + \nu dt + \phi d\zeta, \tag{32}$$

and as such X may become negative. When the deficit reaches a significant magnitude so as to be larger than the realisable value of the firm's assets forced liquidation occurs. In states where liquidation occurs the investment rights to Projects A and B are sold for $F^u(\gamma_A A)$ and $F^u(\gamma_B B)$ respectively.[11] This is the same condition used in the single project case as the buyer of the individual project rights will be an unconstrained firm.

[11]Forced liquidation occurs when $X + G + F^u(\gamma_A A) + F^u(\gamma_B B) \leq 0$

2.3.1 The Unconstrained Firm

The solution to the unconstrained case is an amalgamation of two single project solutions. Because the firm has sufficient cash to invest in either project in any state there will be no interdependence between the projects. The option value to the firm will be

$$F^U(X,A,B) = \begin{cases} (V_A - I_A) + (V_B - I_B) & \text{if } A \geq \hat{A}^u \quad \& \quad B \geq \hat{B}^u \\ (V_A - I_A) + (V_{\hat{B}^u} - I_B)\left(\frac{V_B}{V_{\hat{B}^u}}\right)^{\beta_B} & \text{if } A \geq \hat{A}^u \ \& \ B < \hat{B}^u \\ (V_{\hat{A}^u} - I_A)\left(\frac{V_A}{V_{\hat{A}^u}}\right)^{\beta_A} + (V_B - I_B) & \text{if } A < \hat{A}^u \ \& \ B \geq \hat{B}^u \\ (V_A - I_A)\left(\frac{V_A}{V_{\hat{A}^u}}\right)^{\beta_A} + (V_B - I_B)\left(\frac{V_B}{V_{\hat{B}^u}}\right)^{\beta_B} & \text{if } A < \hat{A}^u \ \& \ B < \hat{B}^u \end{cases} \tag{33}$$

where

$$\hat{A}^u = \frac{\beta_A I_A \delta_A}{\beta_A - 1} \tag{34}$$

and

$$\hat{B}^u = \frac{\beta_B I_B \delta_B}{\beta_B - 1}. \tag{35}$$

2.3.2 The Constrained Firm

The constrained firm does not have the luxury of being able to invest whenever it wants. Instead, there will exist many states where the firm must choose between projects. Investment in a given project will occur if the cash flow from that project is above some minimum threshold, $\hat{A}(X,B)$ and $\hat{B}(X,A)$ for Projects A and B respectively. The investment thresholds of each project are non constant and depend on the cash balance and the potential cash flows of the other project. The payoff to exercising will now depend on which (or both) project is being launched.[12] When investing in both projects simultaneously the option value is the combination of both exercise payoffs

$$(V_A - I_A) + (V_B - I_B). \tag{36}$$

Upon investment in A and delay in B the option will be worth the exercise payoff of A plus the option value of B given that A has been launched and I_A has been spent,

$$(V_A - I_A) + F^{BA}(X - I_A, A, B). \tag{37}$$

The reverse is true for the payoff to investment in B and delay in A, namely the exercise payoff of B plus the option to invest in A given that B has been launched and I_B has been spent

$$(V_B - I_B) + F^{AB}(X - I_B, A, B). \tag{38}$$

It is worth noting that divestment is not relevant in this case because neither project has been launched. Prior to investment the option value will satisfy the following partial differential

[12] In addition to exercising the options the firm may, in any state, voluntarily liquidate and sell the rights to project A and B for $F^u(\gamma_A A)$ and $F^u(\gamma_B B)$ respectively.

equation

$$\frac{1}{2}\sigma_A^2 A^2 F_{AA}+\frac{1}{2}\sigma_B^2 B^2 F_{BB} + (r-\delta_A)AF_A + (r-\delta_B)BF_B + r(X+G)F_X + \frac{1}{2}\phi^2 F_{XX} \\ + AB\rho_{AB}\sigma_A\sigma_B F_{AB} + B\rho_{XB}\sigma_B\phi F_{XB} + A\rho_{XA}\sigma_A\phi F_{XA} - rF = 0 \tag{39}$$

where subscripts on F indicate partial derivatives. The complexity of the equation prevents an analytic solution so numerical methods based on finite differences and successive over relaxation are used to solve it. The following conditions are implemented in the solution procedure. At each investment threshold the option has a value equal to the exercise payoff,

$$F(X, \hat{A}(X,B), B) = (V_{\hat{A}} - I_A) + F^{BA}(X - I_A, \hat{A}, B), \tag{40}$$

$$F(X, A, \hat{B}(X,A)) = (V_{\hat{B}} - I_B) + F^{AB}(X - I_B, A, \hat{B}), \tag{41}$$

$$F(X, \hat{A}(X,B), \hat{B}(X,A)) = (V_{\hat{A}} - I_A) + (V_{\hat{B}} - I_B). \tag{42}$$

When the cash flow of either project hits zero it acts as an absorbing barrier and the project is forever worthless. In such a scenario the problem simplifies down to the one discussed in Section 2.1,

$$F(X, 0, B) = F^B(X, B), \tag{43}$$

$$F(X, A, 0) = F^A(X, A), \tag{44}$$

$$F(X, 0, 0) = 0. \tag{45}$$

In addition, as the cash balance increases, the financing constraint is relaxed and the value of the option will approach that of the unconstrained firm,

$$\lim_{X\to\infty} F(X, A, B) = F^u(A) + F^u(B). \tag{46}$$

As the potential cash flow generated by project A increases, the financing constraint is relaxed and the value of the option will approach that of the unconstrained firm

$$\lim_{A\to\infty} F(X, A, B) = F^u(A) + F^u(B). \tag{47}$$

As the potential cash flow generated by project B increases the financing constraint is relaxed and the value of the option will approach that of the unconstrained firm

$$\lim_{B\to\infty} F(X, A, B) = F^u(A) + F^u(B). \tag{48}$$

Due to the complexity of having two investment opportunities, the firm may now find itself in one of nine situations including liquidation depending on the following investment constraints as well as the liquidation constraint,

$$I_A + I_B \leq X + G + \gamma_A\theta_A V_A + \gamma_B\theta_B V_B, \tag{49}$$

$$I_A \leq X + G + \gamma_A\theta_A V_A, \tag{50}$$

$$I_B \leq X + G + \gamma_B \theta_B V_B. \tag{51}$$

When the first constraint is satisfied, the firm has enough resources to invest simultaneously in both projects. The second and third constraints show when the firm is able to invest in just project A or just project B respectively. It is worth noting that within Equations (50) and (51) the firm does not borrow against one project in order to fund the another. This assumption is consistent with the idea that financial institutions will make lending decisions based on realisable cash flow. When the firm invests in Project A and delays Project B, the bank will not lend any money against Project B as it will not generate any cash until initialised. It is additionally assumed that there are no economies of scale when investing in both projects, to launch both projects will require a lump sump outflow of $I_A + I_B$. The nine states are as follows:

- **Able to invest in both projects simultaneously but not individually**[13]
- **Able to invest in both projects simultaneously as well as individually**[14]
- **Able to invest in both projects simultaneously but only project A individually**[15]
- **Able to invest in both projects simultaneously but only project B individually**[16]
- **Able to invest in either project but not both simultaneously**[17]
- **Only able to invest in project A but not B or both**[18]
- **Only able to invest in project B but not A or both**[19]
- **Forced to delay investment in either project**[20]
- **Forced to liquidate**

It is clear that the addition of another investment opportunity vastly increases the complexity of the investment decision. In order to better understand the investment states and the interaction between them refer to Figure 7, which has been created under the parameters in Table 3. It is worth noting that when $\gamma_B \theta_B V_B < I_B$, the borrowing capacity of project B is less than its investment cost. In this situation, if the firm is able to invest in both projects simultaneously then it must also be such that the firm is able to invest in project A on its own. However, if $\gamma_B \theta_B V_B > I_B$, the borrowing capacity of project B is greater than its investment cost and therefore if the firm is able to invest in project A then it must also be able to do both simultaneously. Consider the graph on the left where $V_B = 96$.[21] In

[13] Equation (49) is satisfied but Equations (50) and (51) are not.

[14] Equations (49), (50) and (51) are satisfied.

[15] Equations (49) and (50) are satisfied but Equation (51) is not.

[16] Equations (49) and (51) are satisfied but Equation (50) is not.

[17] Equations (50) and (51) are satisfied but Equation (49) is not.

[18] Equation (50) is satisfied but Equations (49) and (51) are not.

[19] Equation (51) is satisfied but Equations (49) and (50) are not.

[20] Equations (49), (50) and (51) are not satisfied.

[21] The graphs are shown under $V_B = 96$ and $V_B = 148$ in order to best illustrate the underlying behaviour of the problem.

Table 3: This table outlines the parameter values used to numerically solve Equation (39) for the optimal investment timing policy. The parameters of each project are identical and are where possible similar to those of Boyle and Guthrie (2003).

	Project A	Project B
Project investment cost ($)	$I_A = 100$	$I_B = 100$
Project cash flow volatility	$\sigma_A = 0.20$	$\sigma_B = 0.20$
Project dividend yield	$\delta_A = 0.03$	$\delta_B = 0.03$
Information Asymmetry	$\gamma_A = 0.80$	$\gamma_B = 0.80$
Control friction	$\theta_A = 0.40$	$\theta_A = 0.40$
Riskless interest rate	$r = 0.03$	
Firm cash flow-Project A cash flow correlation	$\rho_{XA} = 0.5$	
Firm cash flow-Project B cash flow correlation	$\rho_{XB} = 0.5$	
Project A cash flow-Project B cash flow correlation	$\rho_{AB} = 0.5$	
Firm cash flow volatility	$\phi = 60$	
Market value of existing assets	$G = 100$	

this case $\gamma_B \theta_B V_B < I_B$ and as such the investment constraint for investing in only project A, as represented by the solid, negative sloping line, binds for lower levels of cash than the constraint for both projects, as represented by the dashed line. This is because in order to do both projects, the firm's net assets (excluding funds raised from using B as collateral) need to be sufficiently greater than I_A in order to make up the shortfall of $I_B - \gamma_b \theta_B V_B$. The investment constraint for launching project B is represented by the vertical line as the firm's ability to invest in B on its own does not depend on A. The curve plots the liquidation constraint. In region 1 the firm has negative net assets and is forced to liquidate. As either cash or V_A increase, the firm moves into Region 2 and their net assets are large enough to avoid forced liquidation however they are still unable to invest. In region 3 the firm is only able to invest in project A. As cash and V_A increase the firm moves into region 4 and can invest in A as well as both projects simultaneously. Here project A is now valuable enough that the extra borrowing capacity associated with it, over and above that required to invest only in A, can be used to subsidise the investment cost of B that otherwise could not be undertaken. Should cash increase even further and the firm find itself in region 5 they will no longer be constrained by opportunities (yet they are still constrained in cash) and can invest in either A or B or both. In region 6 the firm is able to do either A or B but not both. When in region 7 the firm is only able to invest in project B.

Now consider the case where $V_B = 148$ and as such $I_B > \gamma_B \theta_B V_B$. Instead, the investment constraint for both projects now binds for lower levels of cash than the constraint for just A. In regions 1 and 2 the firm is forced to liquidate and wait respectively. In region 3 the firm is forced to do both projects if it chooses to invest. In region 4 the firm can invest in either A or both projects. In region 5 the firm can do either A, B or both projects. For relatively low levels of V_A, as cash decreases the firm moves into region 6 and is able to either B or both projects but not A on its own. In region 7 the firm can only do B.

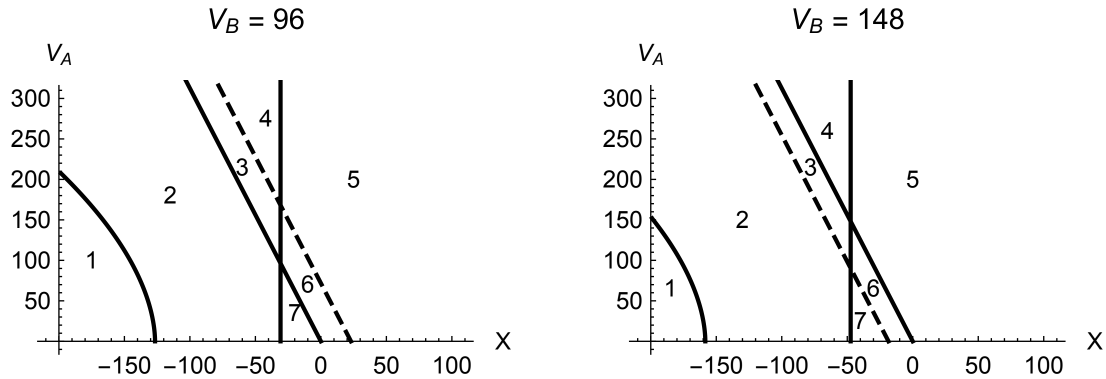

Figure 7: This figure shows some of the regions the constrained firm will find itself in. The sloped black line represents the investment constraint for launching only project A and the sloped dashed black line represents the investment constraint for launching both projects simultaneously. The vertical black line represents the constraint for investing in only project B and the remaining black curve is the liquidation constraint. For the graph on the left: In Region 1, the firm is forced to liquidate. In region 2, the firm is solvent but unable to invest in any capacity. In region 3, the firm can invest in project A only. In region 4, the firm can invest in both projects simultaneously but only A individually. In region 5, the firm can invest in both projects either simultaneously or individually. In region 6, the firm can invest in either project but not both. In region 7, the firm can only invest in project B. For the graph on the right: In region 1, the firm is forced to liquidate. In region 2, the firm is solvent but unable to invest in any capacity. In region 3, the firm can invest in both projects simultaneously but neither individually. In region 4, the firm can invest in both projects simultaneously but only a individually. In region 5, the firm can invest in both projects either simultaneously or individually. In region 6, the firm both projects simultaneously but only B individually. In region 7, the firm can only invest in project B.

The addition of an extra project to the model of Boyle and Guthrie (2003) is a fairly simple extension. However, it greatly increases the number of decisions the firm is able to make and the situations it may find itself in. The following section illustrates the resulting optimal investment policy for the constrained firm.

3 Two Project Investment Policies

The constrained firm will at times find itself in states where although it would like to invest it lacks the capital to do so. When faced with similar investment opportunities the unconstrained firm will undertake investment when it is optimal, as such

$$F(X, A, B) \leq F^u(A) + F^u(B). \tag{52}$$

When subject to the financing frictions as described earlier, the constrained firm will either postpone (under invest) or accelerate (over invest) in a specific project when compared to the benchmark case of the unconstrained firm. Keeping in line with previous analysis, the investment policy of project A will be explored in detail, although the logic can easily be applied to that of project B. Figure 8 displays the optimal investment policy for the unconstrained firm for a variety of V_B. First consider the case where $V_B = 0$. In this situation the firm will invest in project A for any combination of X and V_A that lies above the red line.

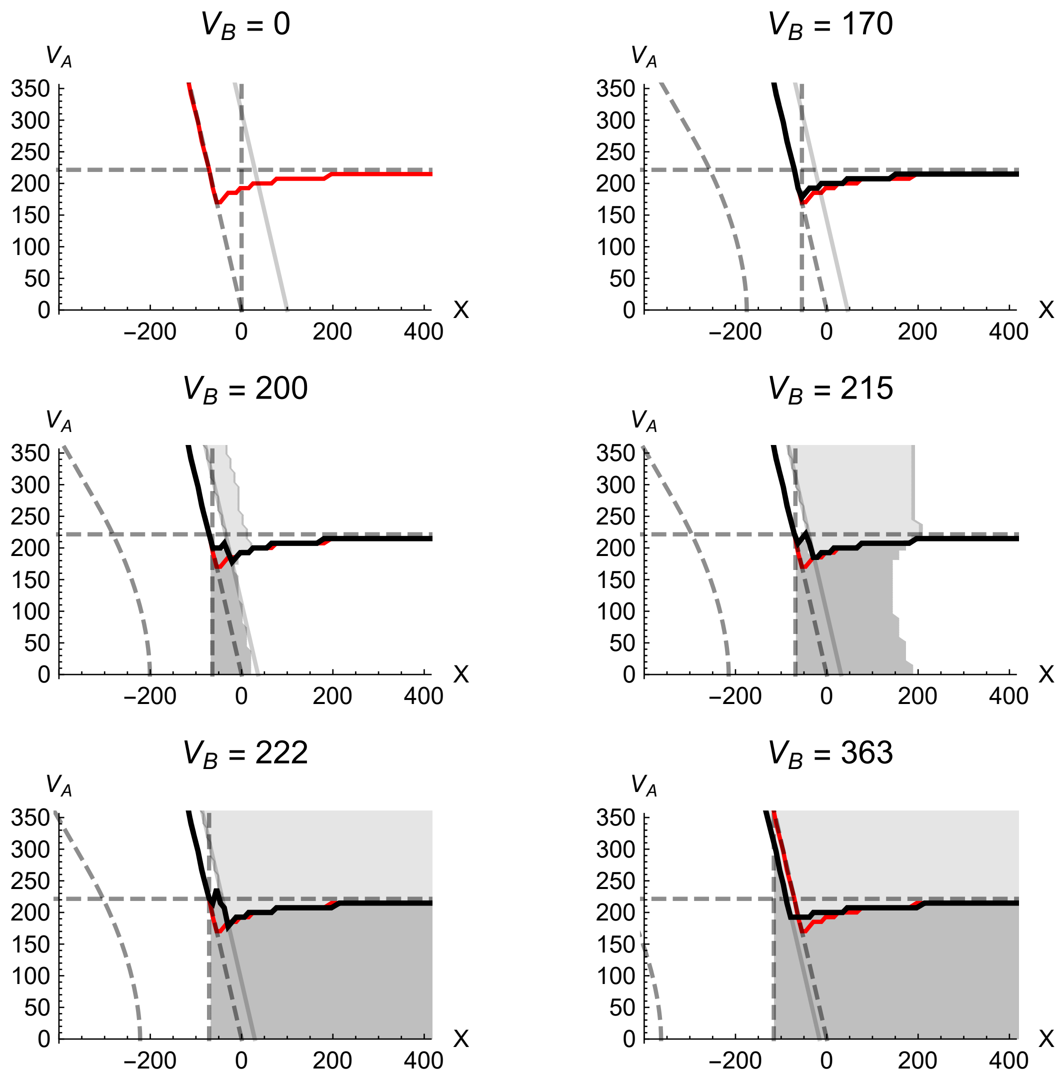

Figure 8: This graph shows the optimal constrained investment policy for a variety of V_B under the parameters in Table 3. The red curve is the single project investment threshold for project A. The black curve is the investment threshold for project A given that the firm also has access to project B. The dark grey region shows states where project B is launched but not project A and the light grey region shows states where both projects are launched simultaneously.

This is the same result as described earlier where the firm has access to only one project. For low levels of cash the investment constraint is binding and the firm invests as soon as possible to avoid future shortfalls. As cash increases and the investment constraint no longer binds the firm reduces the level of investment acceleration and the threshold approaches that of the unconstrained firm.

The effect of having an extra investment opportunity on the policy of project A can be seen in the remaining graphs. When $V_B = 170$ the firm does not change its behaviour very much. The investment constraint still binds to the left where the risk of future funding shortfalls is largest. However, for levels of cash where the constraint does not bind, investment is delayed slightly. The inclusion of B makes the firm delay A slightly longer because investment is irreversible. Launching A will reduce the cash balance by I_A. Therefore, if

B turns out to be a good project ex-post the firm would rather have the cash available to invest than have to wait and build up reserves. It is worth noting that when $V_B = 170$, for the range of X and V_A shown the firm does not invest in B. In this case the delay is entirely caused by the fact the firm may want to do B in the near future.

Eventually V_B will be of such a magnitude that A is delayed because the firm chooses instead to invest in B. Refer to the graph where $V_B = 200$ for a scenario where this happens. In this case the dark grey region represents states where investment in B occurs and the light grey region where investment in both A and B occurs. Here it can be seen that the investment constraint for doing only A still binds for low cash. However, for levels of cash that are just below zero the existence of project B causes investment in A to be considerably delayed. Examining the dark grey region shows that for low cash and low V_A the firm invests in B as soon as possible. This is the same story as before where the firm invests now in order to avoid losing the opportunity to invest in the future. The project choice question is easily answered as B is considerably better than A. As V_A increases, and the firm moves into the dark grey area above the single project threshold (red) and below the multi-project threshold (black) the firm still chooses B. The inclusion of B now causes a large adjustment in the policy for A. Cash is low enough that the firm must invest in something, else it risks losing the ability to invest in anything in the future. In this case when $V_A < V_B = 200$ the firm rightfully chooses B and further delays A. If $V_A > V_B = 200$ then instead it does the opposite and delays B. As cash increases from this area, the threshold to do both investments simultaneously binds and the delay in A relative to the single project reduces until it no longer exists. In this case the choice between investments is no longer the driving factor in the A policy. Instead both projects are launched as soon as possible as they are both relatively valuable. This is similar to earlier where investment occurs because the firm does not want to lose options in the future. However, it is now a matter of losing the ability to do both that drives the decision. For high levels of cash, B is delayed for the entire range of V_A as the likelihood of not being able to invest is low.

As V_B increases further the grey regions expand as waiting is less valuable for higher levels of cash due to the high value of B. When $V_B = 215$ the investment policy for A displays the same behaviour as when $V_B = 200$ however the delay is slightly larger when compared to the single project case. Consider the right hand boundary of the region where the firm only invests in B. When $V_A = 0$, B will be launched up until cash is just below 200. As V_A increases, this boundary shifts inwards. This occurs because as the value of A increases, the firm is willing to wait longer before deploying B to make sure it picks the correct investment. When V_A reaches a moderate level this effect reverses and the dark grey boundary shifts to the right. Now, both projects are sufficiently valuable so the firm does not need to delay B because it is likely they will still be able to do A in the near future. When $V_B = 222$ the firm invests in B as soon as possible and for the entire range of cash shown.

As soon as $\gamma_B \theta_B V_B > I_B$, the "just-A" and "both" constraints cross over and the investment policy returns to the u-shape as first described in Boyle and Guthrie (2003). In such a scenario V_B is large enough that it can be funded entirely by its own borrowing capacity.[22] This, in combination with the fact that B is launched as soon as possible, means that effectively project B just increments the firm's assets in place. That is to say, B has a similar

[22]So long as the value of the firm's remaining assets is not too low.

effect to that associated with increasing G in the single project model.

The standard NPV rule for investment states that if a project has a positive NPV then it should be undertaken as it provides a net increase in value for the firm. Standard real-options analysis shows that there are many states where the value to waiting is greater than the NPV of a project, even when the NPV is positive, and therefore opposes the standard NPV rule. The addition of an exogenous financing constraint in Boyle and Guthrie (2003) further sophisticates the standard real-options argument by showing that due to the risk of future funding shortfalls the constrained firm may invest earlier than its unconstrained counterpart. Additionally, when the firm is sufficiently constrained it may invest later than the unconstrained firm due to a lack of sufficient resources for investment. Extending their model to include two investment opportunities provides even further insights into the constrained investment decision. There still exists states where (1): the firm accelerates investment for fear of losing the ability to do so in the future and (2): the firm delays investment due to insufficient funds; however, it also shows that the firm will delay investment in a specific project due to other investment opportunities. This occurs either due to the fact that (1): investment in the other project reduces resources such that the firm no longer has enough resources for the investment in question and must therefore wait or (2) although investment in the other project has not occurred, it may occur in the near future and therefore the firm delays the project in question in order to avoid losing the ability to do the other project in the future.

In general, the introduction of another investment opportunity leads the firm to delay investment in their current opportunity because they want to make sure they choose the right one. This choice decision is relatively simple under the baseline parameters as the projects are symmetric is all aspects other than present value. It has been shown that investment policies are distorted towards the more valuable project. The following sections introduces projects that are asymmetric in nature and shows that delay relative to the single project situation is not always optimal.

3.1 Fast Payback Projects

The payback period of an investment measures the time it takes for that investment to generate a cumulative amount of cash equal to the initial cost of investment. Traditionally payback period as an investment analysis tool has been doubted as it does not consider the cash flows after payback (Boardman et al., 1982). However, in general, the sooner a project is able to pay back its initial investment cost, the sooner it can begin to generate cash that can help relax the financing constraints of a firm. Boyle and Guthrie (2006) discuss the merits of payback as a capital budgeting technique and show that longer payback projects will require a larger NPV to justify investment. However, their argument is based around the option value of waiting and not driven by a financing constraint. Fast-payback projects can offer benefits over and above their NPV as the cash they generate helps to relax future financial constraints.

In the context of our investment choice model a high payback project will be one that has a high implicit yield as represented by δ_A and δ_B for projects A and B respectively. To investigate how the addition of an extra investment opportunity influences the optimal timing policy of a high yield project the parameters found in Table 4 are used to solve the

Table 4: This table outlines the parameter values used to numerically solve Equation (39) for the optimal investment timing policy when project A has a quick payback period. The parameters are the same as those shown in Table 3 except the implicit dividend yield of project A has been increased from $\delta_A = 0.03$ to $\delta_A = 0.06$.

	Project A	Project B
Project investment cost ($)	$I_A = 100$	$I_B = 100$
Project cash flow volatility	$\sigma_A = 0.20$	$\sigma_B = 0.20$
Project dividend yield	$\delta_A = 0.06$	$\delta_B = 0.03$
Information Asymmetry	$\gamma_A = 0.80$	$\gamma_B = 0.80$
Control friction	$\theta_A = 0.40$	$\theta_A = 0.40$
Riskless interest rate	$r = 0.03$	
Firm cash flow-Project A cash flow correlation	$\rho_{XA} = 0.5$	
Firm cash flow-Project B cash flow correlation	$\rho_{XB} = 0.5$	
Project A cash flow-Project B cash flow correlation	$\rho_{AB} = 0.5$	
Firm cash flow volatility	$\phi = 60$	
Market value of existing assets	$G = 100$	

investment timing model. These parameters are the same as detailed in the symmetric case of the previous section with the exception of the dividend yield of project A. Instead the yield of this project has been doubled from $\delta_A = 0.03$ to $\delta_A = 0.06$. Refer to Figure 9, which plots the new policy results under these parameters. The red curve is the single project investment threshold for project A. The black curve is the investment threshold for project A given that the firm also has access to project B. The dark grey region shows states where project B is launched but not project A and the light grey region shows states where both projects are launched simultaneously.

When $V_B = 0$, the standard single project result applies and the investment threshold for project A is U-shaped in cash reserves.[23] Relative to the unconstrained firm, investment is accelerated for moderate to high levels of cash in order to avoid losing the opportunity to invest. Investment is delayed for low levels as the firm invests as soon as possible. As V_B increases to 133 there are minor changes in the investment policy. For moderate levels of cash, investment is fractionally delayed when compared to the case of $V_B = 0$. This occurs as the firm is willing to delay investment slightly in order to avoid the case where cash is spent on project A and shortly afterwards project B is highly valuable but unobtainable.

The traditional U-Shaped investment threshold no longer holds as V_B increases further and reaches a level of 193. Now V_B is of such value that the constrained firm will launch project B in some states. For a moderate level of cash and low V_A, project B is launched as soon as possible, as indicated by the dark grey region. However for the same moderate level of cash when V_A is high, instead project A is launched. Relative to the single project case, the timing delay of project A is largest in this moderate range of cash. Here the firm is

[23]Project B generates cash flow that follows a GBM process and therefore if $B = 0$ now, then at all times in the future $B = 0$ and the firm can be treated as if it only has access to project A.

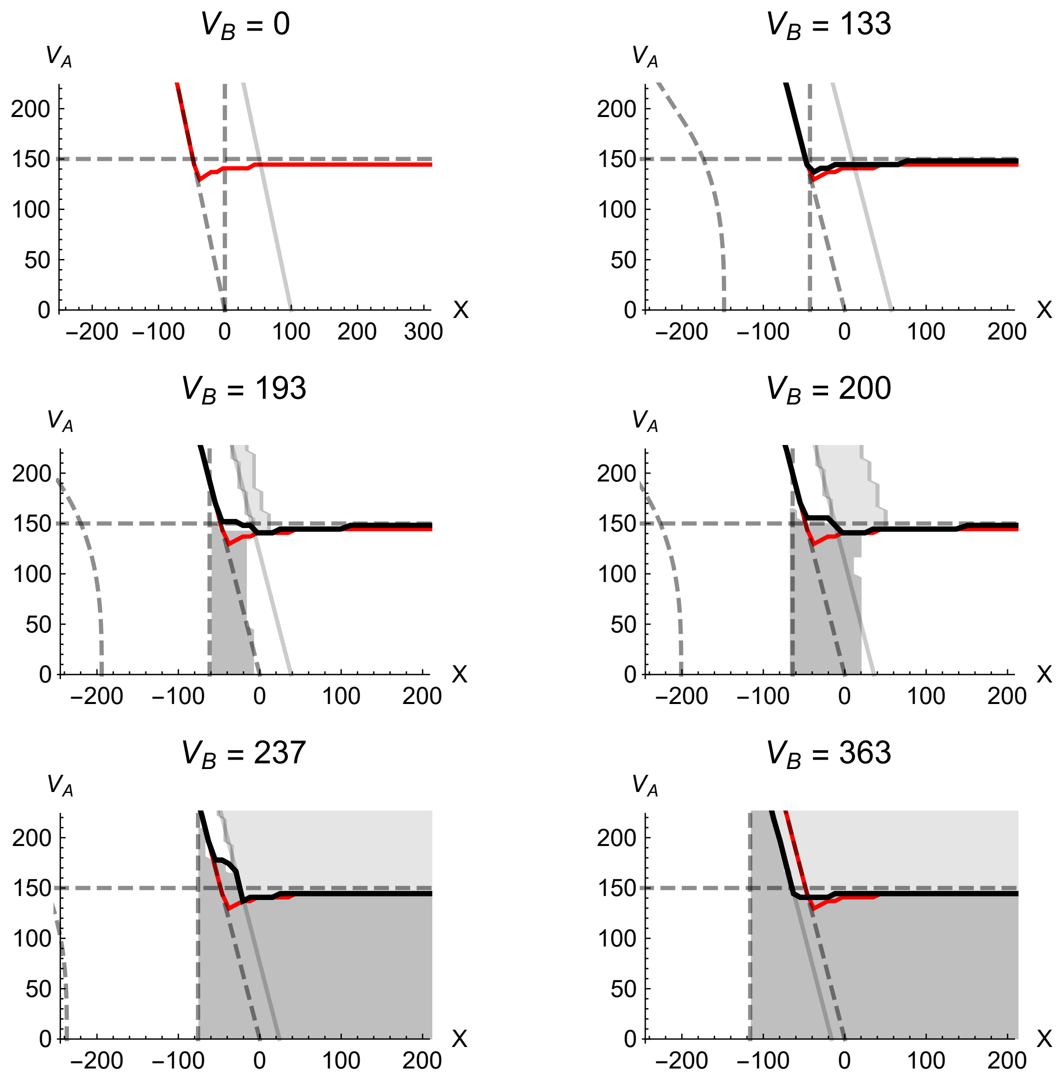

Figure 9: This figure shows the investment policy when project A is a fast-payback project under the parameters in Table 4. The red curve is the single-project investment threshold for project A. The black curve is the investment threshold for project A given that the firm also has access to project B. The dark grey region shows states where project B is launched but not project A and the light grey region shows states where both projects are launched simultaneously.

sufficiently constrained and therefore unable to do both projects and therefore initiates the project that is more valuable. It is worth noting that an intermediate region exists within the area above the single project (red curve) and below the multi-project (black curve) policies for project A. Within this region in some cases the firm chooses to delay investment in both projects. This is in contrast to the symmetric case where the firm implements a strict "A or B" policy and chooses the best project without delay. The delay seen in Figure 9 is driven by the increased dividend yield of project A. As V_A increases, instead of investing in B up until hitting the A threshold the firm will stop investing in B when V_A is sufficiently large and wait to see whether V_A will increase in the future. In this case the loss of potential cash that would have been generated by B is offset by the potential for larger magnitudes of cash to be generated by A.

As V_B increases further and reaches 200, the resulting policies maintain the general shape as above. However, the small interim waiting region described earlier is almost non-existent. It is worth noting that in the moderate range of cash, the policy switch from project B to project A occurs when V_A is around 155. When projects are symmetric in Figure 8 and $V_B = 200$, the switch occurs when both investments have the same present value, that is when V_A is also equal to 200. This is logical as the two projects have the same characteristics, their only point of differentiation being the current cash flow generated and therefore the present value of each project. As such it is simple to see that the firm chooses to invest in the project with the high NPV as neither project provides significant benefits over and above its NPV. However, when project A has a higher yield, the switch will occur when $V_A < V_B$. This shows that when the constrained firm has a level of cash that puts it close to the investment constraints, in some states, it will be willing to choose project A over project B even though project A has a lower NPV. This is in direct violation of the standard NPV rule and occurs because the high yield project is able to provide additional benefits, over and above its NPV. These benefits come in the form of its ability to relax the future financial constraints on follow-up investment. This result supports and provides further insight into the intuition described in Thakor (1990) and Almeida et al. (2011) where firms will have a preference for short payback projects that can relax future financial constraints.

The behaviour described above persists as V_B increases further. Eventually V_B is so large that investment occurs regardless of state and the policy for project A returns to the U-shape of Boyle and Guthrie (2003).[24] Interestingly, and contrary to intuition, for all levels of cash, the introduction of project B for a firm that holds the higher yielding project A does not cause an acceleration of investment in project A relative to if the firm only had project A. This would suggest that although in some states project A is picked over project B, the effect of having cash in the future is outweighed by the cost of spending cash now.[25]

3.2 High-Pledgeability Projects

The pledgeability of an asset refers to the amount of the asset's value that can be used as collateral for debt. Similar to a fast payback project, a project with high pledgeability can provide benefits over and above its NPV as it can raise relatively more capital based on its NPV thus freeing up scarce cash reserves for other projects. The model's financing frictions include an assumption that the collateral value of an asset is lower than the true value of that asset. Asset pledgeability in the model is represented through the financing frictions γ_A, γ_B, θ_A and θ_B. The level of debt that can be raised on an uninitiated project is equal to $\gamma_A \theta_A V_A$ and $\gamma_B \theta_B V_B$ for projects A and B respectively. Therefore, the higher the parameter value the higher the pledgeability of the project. Pledgeable assets support more borrowing, and in turn facilitate further investment in other assets (Almeida and Campello, 2007) and therefore should be preferred by the constrained firm.

To investigate how the addition of an extra investment opportunity influences the optimal timing policy of a high-pledgeability project, the parameters found in Table 5 are used to solve the investment timing model. These parameters are the same as detailed in the symmetric case of the previous section with the exception of the information asymmetry and

[24] As discussed earlier this is because project B is so large that it effectively becomes equivalent to assets

Table 5: This table outlines the parameter values used to numerically solve Equation (39) for the optimal investment timing policy when project A is a high pledgeability project. The parameters are the same as shown in Table 3 except the information asymmetry of project A has changed from $\gamma_A = 0.8$ to $\gamma_A = 0.95$ and the control discount has changed from $\theta_A = 0.4$ to $\theta_A = 0.95$.

	Project A	Project B
Project investment cost ($)	$I_A = 100$	$I_B = 100$
Project cash flow volatility	$\sigma_A = 0.20$	$\sigma_B = 0.20$
Project dividend yield	$\delta_A = 0.03$	$\delta_B = 0.03$
Information Asymmetry	$\gamma_A = 0.95$	$\gamma_B = 0.80$
Control friction	$\theta_A = 0.95$	$\theta_B = 0.40$
Riskless interest rate	$r = 0.03$	
Firm cash flow-Project A cash flow correlation	$\rho_{XA} = 0.5$	
Firm cash flow-Project B cash flow correlation	$\rho_{XB} = 0.5$	
Project A cash flow-Project B cash flow correlation	$\rho_{AB} = 0.5$	
Firm cash flow volatility	$\phi = 60$	
Market value of existing assets	$G = 100$	

control frictions of project A. Instead the information asymmetry friction of this project has been decreased from $\gamma_A = 0.8$ to $\gamma_A = 0.9$ and the control friction decreased from $\theta_A = 0.4$ to $\theta_A = 0.9$. Refer to Figure 10 which plots the new policy results under these parameters. As before, when V_B is equal to zero the standard single project policy shape applies and a U-shaped investment threshold is shown. Depending on the level of cash and project value the firm will either accelerate or delay investment relative to the unconstrained firm. It is worth noting that for high project values the gap between the liquidation constraint and investment constraint is small. This occurs because when V_A is large, in liquidation the buyer of the project will pay $\gamma_A V_A - I_A$ and therefore the liquidation constraint becomes $X + G + \gamma_A V_A - I_A \leq 0$. The investment constraint is $I_A \leq X + G + \gamma_A \theta_A V_A$ and as such the size of the forced waiting region is determined by the control discount θ_A. A high control discount (low parameter value) increases the waiting region and a low discount (high parameter value) reduces it. As V_B increases to 141, the investment policy maintains the U-shape; however, investment is marginally delayed. In this case the delay occurs as the firm does not want to invest in project A to find out that it would rather launch project B ex-post.

Further increasing V_B until $V_B = 178$ leads to a change in the shape of the investment policy. Instead of the traditional U-shape, the investment policy for project A now takes on a distinct W-shape. When the firm has low cash reserves, the investment constraint is binding and investment in A is undertaken as soon as possible. Eventually, cash reaches a sufficient level where the likelihood of losing the ability to invest in the future is small enough that the investment policy jumps upwards quickly with X. In this case, due to the

in place.

[25] In this case, cash in the future refers to cash that would be generated by Project A if it was launched.

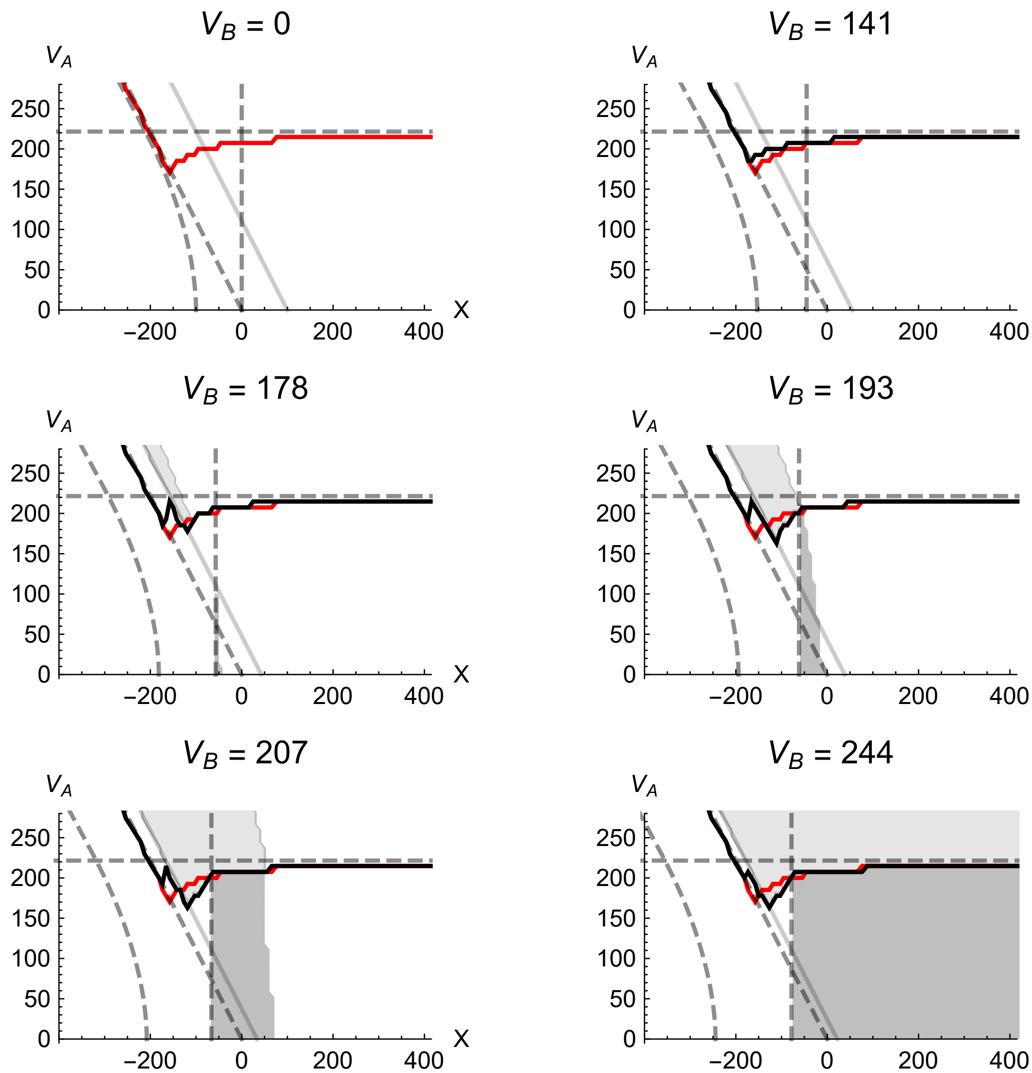

Figure 10: This figure shows the investment policy when project A is a high pledgeability project under the parameters in Table 5. The red curve is the single project investment threshold for project A. The black curve is the investment threshold for project A given that the firm also has access to project B. The dark grey region shows states where project B is launched but not project A and the light grey region shows states where both projects are launched simultaneously.

comparatively low pledgeability of B, the firm is considerably far away from being able to invest in project B. This means that the only feasible way investment in B will occur is through the help of project A. If the firm is unable to invest in both projects simultaneously then should the firm invest in A on its own there are three ways the constraint on B as a follow up investment is relaxed. First, project A will generate cash that can be used to fund future investment. Second, a proportion of the asset can be sold to raise capital that can be used to finance investment. Third, the firm may raise debt on a proportion of the project provided that proportion has not been sold. The dividend yield is not particularly large so the effect of cash generated is negligible, an idea that will be explored further in the following section. Additionally, it is costly for the firm to invest in project A and subsequently sell off part of the project as a capital-raising strategy. Debt financing can be raised at any time for the firm. Therefore, the firm chooses to delay investment in A until such a time when V_A is sufficiently large to facilitate simultaneous investment in both projects. In this case the upwards jump in threshold occurs as a result of the difficulty surrounding investment in project B.

Following the threshold jump, the policy function binds to the simultaneous-investment constraint as cash increases. This leads to the investment threshold of project A decreasing in cash. For higher and higher levels of cash the firm is able to invest in both projects for a lower and lower value of V_A. Here the firm is launching both projects as soon as possible due to the fear of losing the ability to do both in the future. Eventually the simultaneous-investment constraint no longer binds and the firm begins to reduce the level of acceleration relative to the unconstrained firm as cash increases, thus completing the W-shaped investment threshold of project A. In regards to investment in project B it is worth noting that when $V_B = 178$, in most states where B is initiated it is done so in conjunction with project A. There is only a small set of states where project B is launched and project A delayed, which can be seen along the financing constraint for project B.

When V_B is equal to 193 the W-shaped investment curve persists. There are now more states where investment in B occurs as soon as possible without the initiation of project A. Interestingly, where the simultaneous-investment constraint binds the policy shifts from being above the single project policy to below it.[26] Here the introduction of an additional project opportunity can cause the constrained firm to invest earlier in a highly pledgeable project than they would otherwise. This result is consistent with the work of Almeida et al. (2011), who suggest that investment policies will distort towards collteralizable projects. The model used however extends this result to show that the level of distortion, and states where distortion occur, will be highly dependent on the firm's cash reserves and financial position.

The W-shape is maintained in the remaining graphs of Figure 10 as V_B increases to 207 followed by 244. It is worth noting that the policy acceleration relative to the single project case is seen only in the region where project B is unobtainable on its own. Intuitively, this makes sense as it is in this region where the firm needs to use the resources of project A to "cross subsidise" the investment in project B. Acceleration would require the firm to be in a state where they choose to not invest in project A if it was all they had.

4 Concluding Remarks

It is a well known fact that financing constraints restrict the investment decisions available and can therefore lead to a suboptimal investment policy when compared with firms that do not face such constraints. Boyle and Guthrie (2003) use an endogenous liquidity constraint to show that the constrained firm can either invest earlier or later than the unconstrained firm depending on the constrained firm's cash balance. Early investment occurs when the risk of future funding shortfalls (and therefore losing the ability to invest) outweigh the benefits of waiting. Delayed investment occurs when the firm does not have sufficient resources to invest and is forced to wait until such a time that it does. The addition of a second investment opportunity to the model of Boyle and Guthrie (2003) allows the model to not only explore

[26] Although much smaller, this region is also seen when $V_B = 178$.

the constrained investment timing decision but it also provides insights into how the choice *between* multiple projects is made. The principle conclusions outlined within this paper are as follows:

1. Due to information asymmetry, the constrained firm with insufficient resources to invest will be resistant to selling existing assets to fund investment and as such will willingly delay investment past the unconstrained optimum.
2. For the constrained firm with access to one project, the introduction of a second project with symmetrical characteristics will raise the threshold required to invest in the first project as the firm does not want to invest in one project to find out ex-ante that it would rather have invested in the other (or find itself unable to invest in the remaining project in the future although it would like to).
3. A project with a higher implicit dividend yield is able to provide benefits over and above its NPV as the cash it generates can relax future financial constraints. When choosing between projects these additional benefits can lead the constrained firm to choose a lower NPV project instead of higher NPV alternatives as it can better facilitate future investment.
4. A project that can pledge a high proportion of its value as collateral for borrowing is able to free up cash reserves for remaining projects with lower collateral capacity. This can shift the threshold for investment in the high pledgeability project lower than if the firm only had access to this single project.

The analysis in this paper has been completed by looking at a cash constrained firm with access to two investment opportunities. It is unlikely that additional investment opportunities would provide material insights into the problem of the constrained firm. However, there are multiple extensions that can be applied to the two project model to gain further insights into the behaviour of the constrained firm. Perhaps the most interesting area for expansion relates to the role of an internal capital market within a cash constrained firm. The spin-off decision can first be made endogenous in order to allow the firm to run the projects separately. This is likely to yield further insights into when an internal capital market is valuable and when it is not. Further, the cash allocation mechanism in a spin-off situation can be enhanced by introducing self-interested divisional managers. In the presence of asymmetric information, a divisional manager must choose how to represent their divisions prospects to the party that allocates budgets. Understating performance is likely to leave "cash on the table" for other divisions to take while overstating performance is likely to have consequences of its own.

A Equation Derivations

A.1 Derivation of Equation (28)

It is assumed that the risks inherent in A, B and X are spanned by the market of existing securities. Specifically, suppose that there are traded assets or portfolios with prices a, b and x that evolve according to

$$dv_a = \mu_a a dt + \sigma_a v d\epsilon_1 \tag{53}$$

$$dv_b = \mu_b b dt + \sigma_b v d\epsilon_2 \tag{54}$$

$$dx = \mu_x x dt + \sigma_x v d\zeta \tag{55}$$

Then a long position in the investment option can be combined with a short position of $\sigma_A A F_A/(\sigma_a a)$ units of asset a, a short position of $\sigma_B B F_B/(\sigma_b b)$ units of asset b and $\phi F_X/(\sigma_x x)$ units of asset x to produce a total return dR over the time interval dt such that

$$dR = dF - \left(\frac{\sigma_A A F_A}{\sigma_a a}\right) da - \left(\frac{\sigma_B B F_B}{\sigma_b b}\right) db - \left(\frac{\phi F_X}{\sigma_x x}\right) dx. \tag{56}$$

Using Ito's Lemma to obtain an expression for dF, substituting (53), (54) and (55) for da, db and dx, respectively, and simplifying, dR becomes

$$\begin{aligned} dR = \Bigg(& \frac{1}{2}\sigma_A^2 A^2 F_{AA} + \frac{1}{2}\sigma_B^2 B^2 F_{BB} + \frac{1}{2}\phi^2 F_{XX} + \rho_{AB}\sigma_A\sigma_B AB F_{AB} + \rho_{XB}\sigma_B\phi B F_{XB} \\ & + \rho_{XA}\sigma_A\phi A F_{XA} + \left(\mu_A - \frac{\mu_a\sigma_A}{\sigma_a}\right) A F_A + \left(\mu_B - \frac{\mu_b\sigma_B}{\sigma_b}\right) B F_B + \left(rX + \nu + A - \frac{\mu_x\phi}{\sigma_x}\right) F_X \Bigg) dt. \end{aligned} \tag{57}$$

It is worth noting that the coefficient on F_X reflects the fact that project A is in operation and generating cash flows. Since this return is risk-free, the portfolio must earn the risk-free rate of return. Therefore,

$$dR = r\left(F - \frac{\sigma_A A F_A}{\sigma_a} - \frac{\sigma_A B F_B}{\sigma_b} - \frac{\phi F_X}{\sigma_x}\right) dt. \tag{58}$$

Equating this to the above expression for dR means that F satisfies the differential equation

$$\begin{aligned} & \frac{1}{2}\sigma_A^2 A^2 F_{AA} + \frac{1}{2}\sigma_B^2 B^2 F_{BB}\frac{1}{2} + \phi^2 F_{XX} + \rho_{AB}\sigma_A\sigma_B F_{AB} + \rho_{XB}\sigma_B\phi F_{XB} + \rho_{XA}\sigma_A\phi F_{XA} \\ & + \left(\mu_A - \frac{\mu_a\sigma_A}{\sigma_a} + \frac{r\sigma_A}{\sigma_a}\right) A F_A + \left(\mu_B - \frac{\mu_b\sigma_B}{\sigma_b} + \frac{r\sigma_B}{\sigma_b}\right) B F_B + \left(rX + \nu + A - \frac{\mu_x\phi}{\sigma_x} + \frac{r\phi}{\sigma_x}\right) F_X - rF = 0 \end{aligned} \tag{59}$$

Further simplification can most readily be obtained by defining

$$\lambda_A = \frac{\mu_a \sigma_A}{\sigma_a} - \frac{r\sigma_A}{\sigma_a} = \frac{\sigma_A}{\sigma_a}\left(\mu_a - r\right),$$
$$\lambda_B = \frac{\mu_b \sigma_B}{\sigma_b} - \frac{r\sigma_B}{\sigma_b} = \frac{\sigma_B}{\sigma_b}\left(\mu_b - r\right) \tag{60}$$

where λ_A and λ_B are the risk premiums of project A and project B respectively, $\mu_a - r$ and $\mu_b - r$ the risk premiums of the spanning assets a and b respectively and σ_A/σ_a and σ_B/σ_b adjust for the difference in volatilities between projects and spanning assets. If $\delta_A = r + \lambda_A - \mu_A$ and $\delta_B = r + \lambda_B - \mu_B$ are the dividend yields of project A and B respectively, the (59) coefficient on AF_A, $\mu_A - (\mu_a\sigma_A/\sigma_a) + (r\sigma_A/\sigma_a)$, becomes $r - \delta_A$ and the (59) coefficient on BF_B, $\mu_B - (\mu_b\sigma_B/\sigma_b) + (r\sigma_B/\sigma_b)$, becomes $r - \delta_B$.

Now let G denote the market value of a claim to the future cash flow generated be the firm's existing physical assets excluding project A. Clearly G is independent of X, A and B, so $dG = 0$ over any time interval dt. Thus, the return on a long position in G consists only of the current cash flow ($\nu dt + \phi d\zeta$). Hence, using (55), a long position in G combined with a short position in $\phi/(\sigma_x x)$ units of asset x yields a total return of

$$\nu dt + \phi d\zeta - \left(\frac{\phi}{\sigma_x x}\right) dx = \left(\nu - \frac{\phi\mu_x}{\sigma_x}\right) dt. \tag{61}$$

Since this return is risk-free, it must be that

$$\nu - \frac{\phi\mu_x}{\sigma_x} = r\left(G - \frac{\phi}{\sigma_x}\right), \tag{62}$$

which implies that the (59) coefficient on F_X, $rX + \nu - (\mu_x\phi/\sigma_x) + (r\phi/\sigma_x)$, is equal to $r(X + G) + A$. Making this substitution back into (59) yields Equation (28). The same derivation above applies for Equation (39) with A omitted from the coefficient on F_X.

B Solution Procedures

B.1 Numerical Solution Procedure for Equation (11)

The partial differential equation is solved on a grid with nodes

$$\{(X_l, A_m) : l = 1, \ldots, L, m = 1, \ldots, M\} \tag{63}$$

where $X_l - X_{l-1} = dX$ and $A_m - A_{m-1} = dA$. The investment and liquidation constraints are

$$I_A \leq X + G + \gamma_A \theta_A V_{A_m} \tag{64}$$

$$X + G + F^{UA}(\gamma_A A_m) \leq 0 \tag{65}$$

respectively where $V_{A_m} = A_m/\delta_A$. The respective investment and liquidation payoffs are $V_{A_m} - I_A$ and $F^{UA}(\gamma_A A_m)$. At node (X_l, A_m) the resulting difference equation can be written in the form

$$\begin{aligned} 0 =& a_m F_{l,m+1} + b_m F_{l,m} + c_m F_{l,m-1} + d_l F_{l+1,m} + e_l F_{l-1,m} \\ &+ f_m (F_{l+1,m+1} - F_{l+1,m-1} - F_{l-1,m+1} + F_{l-1,m-1}) \end{aligned} \tag{66}$$

where

$$\begin{aligned} a_m &= \frac{\sigma_A A_m^2}{2dA^2} + \frac{(r - \delta_A)A_m}{2dA} \\ b_m &= -\frac{\sigma_A A_m^2}{dA^2} - \frac{\phi^2}{dX^2} - r \\ c_m &= \frac{\sigma_A A_m^2}{2dA^2} - \frac{(r - \delta_A)A_m}{2dA} \\ d_l &= \frac{r(X_l + G)}{2dX} + \frac{\phi^2}{2dX} \\ e_l &= -\frac{r(X_l + G)}{2dX} + \frac{\phi^2}{2dX^2} \\ f_m &= \frac{\rho_{XA}\sigma_A \phi A_m}{4dXdA} \end{aligned} \tag{67}$$

and $F_{l,m} = F(X_l, A_m)$. This equation is defined whenever $2 \leq l \leq L-1$ and $2 \leq m \leq M-1$. It is extended to the edges of the grid using four boundary conditions: (i) When $l = 1$, if the investment constraint is satisfied then $V_{A_m} - I_A$ is used, if the liquidation constraint is satisfied then $F^{UA}(\gamma_A A_m)$ is used and for all other nodes

$$F^{UA}(\gamma_A A_m) + \frac{X_1(V_{A_m} - I_A - F^{UA}(\gamma_A A_m))}{I_A - G - \gamma_A \theta_A V_{A_m}}, \tag{68}$$

a linear interpolation between the investment and liquidity regions is used. (ii) When $l = L$, $F^{UA}(A_m)$, the unconstrained value is used. (iii) When $m = 1$, $F_{l,1} = 0$ is used since $F(X, 0) = 0$. (iv) When $m = M$, if the investment constraint is satisfied then $V_{A_M} - I_A$ is used, if the liquidation constraint is satisfied then $F^{UA}(\gamma_A A_M)$ is used and for all other nodes

$$F^{UA}(\gamma_A A_M) + \frac{X_l(V_{A_M} - I_A - F^{UA}(\gamma_A A_M))}{I_A - G - \gamma_A \theta_A V_{A_M}}, \tag{69}$$

a linear interpolation between the investment and liquidity regions is used.

The solution is begun by setting $F_{l,m} = V_{A_m} - I_A$ if the investment constraint is satisfied as well as $V_{A_m} - I_A > 0$. At all other nodes $F_{l,m} = F^{UA}(\gamma_A A_m)$ is used. After implementing the initial solution the system is solved using the method of Successive Over Relaxation. During each iteration of this method, the difference equation is solved at each node (X_l, A_m) in turn. The calculated value of $F_{l,m}$ is replaced with $F^{UA}(\gamma_A A_m)$ if the liquidation constraint is binding, and with $V_{A_m} - I_A$ if doing so would result in a higher node value as well as the investment constraint being satisfied. Iteration is stopped when the largest change in any $F_{l,m}$ measured relative to its value at the end of the preceeding iteration is less than 0.01.

B.2 Numerical Solution Procedure for Equation (28)

The partial differential equation is solved on a grid with nodes

$$\{(X_l, A_m, B_n) : l = 1, \ldots, L, m = 1, \ldots, M, n = 1, \ldots, N\} \tag{70}$$

where $X_l - X_{l-1} = dX$, $A_m - A_{m-1} = dA$ and $B_n - B_{n-1} = dB$. The firm can invest without divesting assets if

$$I_B \leq X + G + \gamma_A \theta_A V_{A_m} + \gamma_B \theta_B V_{B_n}, \tag{71}$$

invest by divesting assets if

$$I_B \leq X + G + \gamma_A V_{A_m} + \gamma_B \theta_B V_{B_n}, \tag{72}$$

and are forced into liquidation when

$$X + G + \gamma_A V_{A_m} + F^{UB}(\gamma_B B_n) \leq 0. \tag{73}$$

The payoff to investing without divesting is

$$V_{B_n} - I_B, \tag{74}$$

to investing by divesting the optimal amount

$$V_{B_n} - I_B - \left(\frac{J_B}{\gamma_A} - \theta_A V_{A_m}\right)\left(\frac{1-\gamma_A}{1-\theta_A}\right) \tag{75}$$

where $J_B = I_B - X - G - \gamma_B \theta_B V_{B_n}$, and liquidation is

$$F^{UB}(\gamma_B B_n). \tag{76}$$

At node (X_l, A_m, B_n) the resulting difference equation can be written in the form

$$\begin{aligned} 0 = {} & a_m F_{l,m+1,n} + b_{m,n} F_{l,m,n} + c_m F_{l,m-1,n} + d_n F_{l,m,n+1} + e_n F_{l,m,n-1} + f_{l,m} F_{l+1,m,n} \\ & + g_{l,m} F_{l-1,m,n} + h_{m,n}(F_{l,m+1,n+1} - F_{l,m+1,n-1} - F_{l,m-1,n} + F_{l,m-1,n-1}) \\ & + i_m(F_{l+1,m+1,n} - F_{l+1,m-1,n} - F_{l-1,m+1,n} + F_{l-1,m-1,n}) \\ & + j_n(F_{l+1,m,n+1} - F_{l+1,m,n-1} - F_{l-1,m,n+1} + F_{l-1,m,n-1}), \end{aligned}$$

where

$$\begin{aligned} a_m &= \frac{\sigma_A^2 A_m}{2dA^2} + \frac{(r-\delta_A)A_m}{2dA}, \\ b_{m,n} &= -\frac{\sigma_A^2 A_m^2}{dA^2} - \frac{\sigma_B^2 B_n^2}{dB^2} - \frac{\phi^2}{dX^2} - r, \\ c_m &= \frac{\sigma_A^2 A_m^2}{2dA^2} - \frac{(r-\delta_A)A_m}{2dA}, \\ d_n &= \frac{\sigma_B^2 B_n^2}{2dB^2} + \frac{(r-\delta_B)B_n}{2dB}, \\ e_n &= \frac{\sigma_B^2 B_n^2}{2dB^2} - \frac{(r-\delta_B)B_n}{2dB}, \\ f_{l,m} &= \frac{r(X_l+G)+A_m}{2dX} + \frac{\phi^2}{2dX^2}, \\ g_{l,m} &= -\frac{r(X_l+G)+A_m}{2dX} + \frac{\phi^2}{2dX^2}, \\ h_{m,n} &= \frac{\rho_{AB}\sigma_A\sigma_B A_m B_n}{4dAdB}, \\ i_m &= \frac{\rho_{XA}\sigma_A\phi A_m}{4dXdA}, \\ j_n &= \frac{\rho_{XB}\sigma_B\phi B_n}{4dXdB}, \end{aligned} \tag{77}$$

and $F_{l,m,n} = F^{BA}(X_l, A_m, B_n)$. This equation is defined whenever $2 \leq l \leq L-1$, $2 \leq m \leq M-1$ and $2 \leq n \leq N-1$. It is extended to the edges of the grid using six boundary conditions: (i) When $l = 1$, the logic demonstrated in Table (6) is applied. Panel One lists the investment and liquidation constraints and Panel Two the investment/liquidation payoffs. For each row, given that the constraint in Panel One is satisfied the maximum of the selected payoffs in Panel Two is applied. All remaining nodes are set to

$$F^{UB}(\gamma_B B_n) + X_1 \left(\frac{V_{B_n} - I_B - V_{A_m}(1-\gamma_A) - F^{UB}(\gamma_B B_n)}{I_B - G - \gamma_A V_{A_m} - \gamma_B \theta_B V_{B_n}} \right), \tag{78}$$

a linear interpolation between the liquidation and investment via divestment regions. (ii) When $l = L$, $F^{BA}_{L,m,n} = F^{UB}(B_n)$, the unconstrained value. (iii) When $m = 1$, $F^{BA}_{l,1,n} = F(X_l, B_n)$ is used since $F^{BA}(X, 0, B) = F(X, B)$. (iv) When $m = M$, $F^{BA}_{l,M,n} = F^{UB}(B_n)$, the unconstrained value is used. This is because the asset in place is generating sufficient cash to make the firm unconstrained. (v) When $n = 1$, $F^{BA}_{l,m,1} = 0$ is used because $F^{BA}(X, 0, A) = 0$.

(vi) When $n = N$, the logic demonstrated in Table (6) is applied as before. All remaining nodes are set to

$$F^{UB}(\gamma_B B_N) + X_l \left(\frac{V_{B_N} - I_B - V_{A_m}(1-\gamma_A) - F^{UB}(\gamma_B B_N)}{I_B - G - \gamma_A V_{A_m} - \gamma_B \theta_B V_{B_N}} \right), \tag{79}$$

a linear interpolation between the liquidation and investment via divestment regions.

One			Two		
71	72	73	74	75	76
✓			✓		✓
	✓			✓	✓
		✓			✓

Table 6: This table demonstrates the logic applied at the boundaries when $m = M$ and $n = N$ as well as the logic used for the initial solution. Panel One lists the investment constraints as well as the liquidation constraint, Panel Two lists the investment payoffs as well as the liquidation payoff. Working from left to right if the equations in Panel One are satisfied, as illustrated by a tick, the maximum is taken of the selected payoffs in Panel Two. For the boundary conditions all other nodes are set to a linear interpolation between the liquidation and investment regions as described by Equations (88) and (89). While in the initial solution all remaining nodes are set to the liquidation payoff. The table is also applied when Successive Over Relaxation is implemented replacing the PDE solution with the Panel Two results if they are larger conditional on the Panel One Equations holding.

The solution is begun by setting $F_{l,m,n}$ equal to (74) if Equation (71) is satisfied and (76) at all other nodes. After implementing the initial solution the system is solved using the method of Successive Over Relaxation. During each iteration of this method, the difference equation is solved at each node (X_l, A_m, B_n) in turn. The calculated value of $F_{l,m,n}$ is replaced with an investment/liquidation value according to the logic of Table (6) if doing so would result in a higher node value. Iteration is stopped when the largest change in any $F_{l,m,n}$ measured relative to its value at the end of the preceding iteration is less than 0.01. The same method and procedure is used to solve for F^{AB}, the option to invest in project A given that project B is in place and generating cash.

B.3 Numerical Solution Procedure for Equation (39)

The partial differential equation is solved on a grid with nodes

$$\{(X_l, A_m, B_n) : l = 1, \ldots, L, m = 1, \ldots, M, n = 1, \ldots, N\},$$

where $X_l - X_{l-1} = dX$, $A_m - A_{m-1} = dA$ and $B_n - B_{n-1} = dB$. The following constraints are used within the solution procedure:

$$I_A + I_B \leq X_l + G + \gamma_A \theta_A V_{A_l} + \gamma_B \theta_B V_{B_n}, \tag{80}$$

$$I_A \leq X_l + G + \gamma_A \theta_A V_{A_l}, \tag{81}$$

$$I_B \leq X_l + G + \gamma_B \theta_B V_{B_n}, \tag{82}$$

$$X_l + G + F^{UA}(\gamma_A A_m) + F^{UB}(\gamma_B B_n) \leq 0. \tag{83}$$

The first three represent investment constraints while the last one is the liquidation constraint. At node (X_l, A_m, B_n) the payoff to investing in both A and B is

$$V_{A_m} - I_A + V_{B_n} - I_B, \tag{84}$$

where $V_{A_m} = A_m/\delta_A$ and $V_{B_n} = B_n/\delta_B$, the payoff to investing in A and postponing investment in B is

$$V_{A_m} - I_A + F^{BA}(X_l - I_A, A_m, B_n) \tag{85}$$

where $F^{BA}(X_l - I_A, A_m, B_n)$ is an interpolated[27] value of the numerical solution to F^{BA}, the payoff to investing in B and postponing investment in A is

$$V_{B_n} - I_B + F^{AB}(X_l - I_B, A_m, B_n) \tag{86}$$

where and $F^{AB}(X_l - I_B, A_m, B_n)$ is an interpolated[28] value of the numerical solution to F^{AB} and the payoff to liquidation is

$$F^{UA}(\gamma_A A_m) + F^{UB}(\gamma_B B_n). \tag{87}$$

At node (X_l, A_m, B_n) the resulting difference equation can be written in the form

$$\begin{aligned}0 = a_m F_{l,m+1,n} + b_{m,n} F_{l,m,n} + c_m F_{l,m-1,n} + d_n F_{l,m,n+1} + e_n F_{l,m,n-1} + f_l F_{l+1,m,n} \\ +g_l F_{l-1,m,n} + h_{m,n}(F_{l,m+1,n+1} - F_{l,m+1,n-1} - F_{l,m-1,n} + F_{l,m-1,n-1}) \\ +i_m(F_{l+1,m+1,n} - F_{l+1,m-1,n} - F_{l-1,m+1,n} + F_{l-1,m-1,n}) \\ +j_n(F_{l+1,m,n+1} - F_{l+1,m,n-1} - F_{l-1,m,n+1} + F_{l-1,m,n-1}),\end{aligned}$$

[27]Extrapolation is used when $X_l - I_A$ lies outside the grid.
[28]Extrapolation is used when $X_l - I_B$ lies outside the grid.

where

$$a_m = \frac{\sigma_A^2 A_m}{2dA^2} + \frac{(r-\delta_A)A_m}{2dA},$$
$$b_{m,n} = -\frac{\sigma_A^2 A_m^2}{dA^2} - \frac{\sigma_B^2 B_n^2}{dB^2} - \frac{\phi^2}{dX^2} - r,$$
$$c_m = \frac{\sigma_A^2 A_m^2}{2dA^2} - \frac{(r-\delta_A)A_m}{2dA},$$
$$d_n = \frac{\sigma_B^2 B_n^2}{2dB^2} + \frac{(r-\delta_B)B_n}{2dB},$$
$$e_n = \frac{\sigma_B^2 B_n^2}{2dB^2} - \frac{(r-\delta_B)B_n}{2dB},$$
$$f_l = \frac{r(X_l+G)}{2dX} + \frac{\phi^2}{2dX^2},$$
$$g_l = -\frac{r(X_l+G)}{2dX} + \frac{\phi^2}{2dX^2},$$
$$h_{m,n} = \frac{\rho_{AB}\sigma_A\sigma_B A_m B_n}{4dAdB},$$
$$i_m = \frac{\rho_{XA}\sigma_A\phi A_m}{4dXdA},$$
$$j_n = \frac{\rho_{XB}\sigma_B\phi B_n}{4dXdB},$$

and $F_{l,m,n} = F(X_l, A_m, B_n)$. This equation is defined whenever $2 \leq l \leq L-1$, $2 \leq m \leq M-1$ and $2 \leq n \leq N-1$. It is extended to the edges of the grid using six boundary conditions: (i) When $l = 1$, it is supposed that the liquidation constraint is binding, so that $F_{1,m,n} = F^{UA}(\gamma_A A_m) + F^{UB}(\gamma_B B_n)$. (ii) When $l = L$, $F_{L,m,n} = F^{UA}(A_m) + F^{UB}(B_n)$, the sum of the unconstrained values. (iii) When $m = 1$, $F_{l,1,n} = F(X_l, B_n)$ is used since $F(X,0,B) = F(X,B)$. (iv) When $n = 1$, $F_{l,m,1} = F(X_l, A_m)$ is used since $F(X,A,0) = F(X,A)$. (v) Table (7) demonstrates the logic used at the boundary when $m = M$. Panel One lists the three investment constraints and Panel Two the investment payoffs. For each row, given that the constraints in Panel One are satisfied the maximum is taken of the selected payoffs in Panel Two and applied. All remaining nodes are set to

$$F^{UA}(\gamma_A A_M) + F^{UB}(\gamma_B B_n) + \frac{(X_l+G)(V_{A_M} - I_A + F^{BA}(X_l - I_A, A_M, B_n) - F^{UA}(\gamma_A A_M) - F^{UB}(\gamma_B B_n))}{I_A - \gamma_A\theta_A V_{A_M}}, \tag{88}$$

a linear interpolation between the liquidation and investment region. (vi) The same logic of Table (7) is used at the boundary when $n = N$. All remaining nodes are set to

$$F^{UA}(\gamma_A A_m) + F^{UB}(\gamma_B B_N) + \frac{(X_l+G)(V_{B_N} - I_B + F^{AB}(X_l - I_B, A_m, B_N) - F^{UA}(\gamma_A A_m) - F^{UB}(\gamma_B B_N))}{I_B - \gamma_B\theta_B V_{B_N}}, \tag{89}$$

One				Two			
80	81	82	83	84	85	86	87
✓	✓	✓		✓	✓	✓	✓
✓	✓			✓	✓		✓
✓		✓		✓		✓	✓
	✓	✓			✓	✓	✓
	✓				✓		✓
		✓				✓	✓
			✓				✓

Table 7: This table demonstrates the logic applied at the boundaries when $m = M$ and $n = N$ as well as the logic used for the initial solution. Panel One lists the investment constraints as well as the liquidation constraint, Panel Two lists the investment payoffs as well as the liquidation payoff. Working from left to right if the equations in Panel One are satisfied, as illustrated by a tick, the maximum is taken of the selected payoffs in Panel Two. For the boundary conditions all other nodes are set to a linear interpolation between the liquidation and investment regions as described by Equations (88) and (89). While in the initial solution all remaining nodes are set to the liquidation payoff. The table is also applied when Successive Over Relaxation is implemented replacing the PDE solution with the Panel Two results if they are larger conditional on the Panel One Equations holding.

a linear interpolation between the liquidation and investment region. The solution is begun by applying the logic displayed within Table (7) to all applicable nodes in a similar fashion to boundary conditions (v) and (vi). After this process all remaining nodes are set to equal the liquidation payoff. After implementing the initial solution the system is solved using the method of Successive Over Relaxation. During each iteration of this method, the difference equation is solved at each node (X_l, A_m, B_n) in turn. The calculated value of $F_{l,m,n}$ is replaced with an investment/liquidation value according to the logic of Table (7) if doing so would result in a higher node value. Iteration is stopped when the largest change in any $F_{l,m,n}$ measured relative to its value at the end of the preceding iteration is less than 0.01.

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