When Harry Met Kelly: Approximations to Optimal Capital Growth in Markowitz-Tobin Mean-Standard Deviation Space

1 Introduction: optimal investment over time

The standard models of optimal investment – the contingent claims model and the mean-standard deviation model – are based on there being a single period in which to invest. As soon as the dynamics of time is introduced, things do become more interesting, and of course more realistic. In principle, under a discrete time model, an investor would simply look at what is the available fund and the available investment opportunities at the start of each period, and then they would apply a single period model each time. So long as each period is independent of all others, that is fine for an investor who has a standard objective of expected utility maximisation. However, it can be argued that realistically some (perhaps even most) investors might have a different objective, namely that of maximising the long-term growth of their wealth. Interestingly, as is well-known, the analysis of the optimal strategy of such an investor actually reduces straight back down to a case of expected utility maximisation. Concretely it corresponds to the very specific case of expected utility with log-utility. The fact that long-term growth of wealth is maximised by a strategy that mimics that of an expected utility maximiser with log utility has become known as the "Kelly Criterion", due to the fact that the seminal article that foresaw this feature (published in information science, not in finance) was written by John Kelly (see Kelly, 1956). The relevance of the idea for mainstream finance, however, was quickly acknowledged by Latane (1959) and Breiman (1961), and it has more recently been extensively covered in the collection of articles edited by MacLean et. al. (2011). Indeed, the fact that the Kelly Criterion remains of contemporary interest is exemplified by the constant stream of research dealing with the topic in one way or another (see, for example, Lo et al., 2017; Byrnes and Barnett, 2018; Chu et al., 2018; Tran and Verhoeven, 2021; Bermin and Holm, 2023; and Jacot and Mochkovitch, 2023).

An interesting aspect of the literature on growth-maximising investments is to find an approximation to the optimal investment strategy that is described only by the mean and standard deviation of the investments at hand. Thus, an approximation to the true strategy is sought, where the approximation relies on simplifications regarding the statistical characteristics of the available investment opportunities. The best-known such approximation was found by Edward O. Thorp (see, for example, Thorp, 2006), but there are multiple other articles, both by Thorp and by others that derive the same result. Thorp's approximation is that the fraction of wealth that should be invested in the risky option is equal to the Sharpe ratio divided by the standard deviation of the risk. However, that result relies upon an assumption of continuous time, and so the case of the approximate optimal strategy in discrete time is left unanswered.

Of course, continuous time models are important and relevant, but so are discrete time models. There are (at least) two clear justifications for the relevance of discrete rather than continuous time. First, if there is any fixed cost element at all in the portfolio adjustment process, then even if somehow an investor could potentially make continuous micro-adjustments to their portfolios (that is, adjustments made miniscule fractions of seconds apart), the balancing of the costs and benefits of doing so leads naturally to it being optimal for adjustments to be made periodically rather than continuously (see, for example, Goldsmith, 1976, and Wilson, 2016). Depending on the size of the fixed cost of portfolio adjustment, the optimal period size could be intervals of days, hours or even minutes, but it will not be continuous. Second, and perhaps more relevantly, portfolio adjustments are only optimal when new information comes to hand that changes the investor's perception of the parameters involved. However, as has been significantly studied in the literature, a certain degree of investor innatention to news is optimal (or, to use the accepted term, "rational"). Rational innatention was first suggested by Sims (2003), but now has a fairly long history in mainstream finance (see Mackowiak et al., 2023, for a recent survey). Under the theory of rational innatention, even if information flows continuously, a rational investor will only seek updated news on the investment opportunities at discrete points of time. Since portfolios will only be adjusted when new information comes to hand, again the theory of rational innatention leads directly to the appropriate model for portfolio choice being discrete rather than continuous.

In the present article, I reconsider the traditional continuous time approximation as an optimal strategy for an investor with discrete time opportunities for portfolio adjustments. I compare Thorp's approximation with another that has a more natural underpinning for discrete time - the traditional Markowitz mean-standard deviation setting. I show that Thorp's approximation can be improved upon, and that a new approximation may work even better. Here, "better" is defended by comparing the approximation with the first-best solution for a simulated, but quite general, setting.

2 Optimal strategy to maximise the growth of an investment fund with a risky binomial investment

When an investor's objective is to maximise the growth of their wealth (as opposed, for example, to maximising the expected utility of final wealth), then that investor's optimal strategy mimics exactly the optimal strategy of an expected utility maximiser with log utility function for wealth (see, for example, the early articles of Latane 1959 and Breiman 1961). Say the investor contemplates n consecutive investment periods, where in period i a fraction λ_i of wealth available in that period is invested in the risky asset. The investor is interested in designing a strategy, as defined by the vector of investment fractions $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, such that the growth rate of wealth is maximised. Then that investor's optimal strategy is that which would maximise the average of the log of the periodic wealth values. Furthermore, since the log utility function displays constant relative risk aversion and the investment that we are positing is a relative investment, each investment choice λ_i is independent of the actual value of current wealth w_i. Thus, in this investment scenario each λ_i can *only* depend on the periodic asset retrun rates, and not on the actual level of available wealth in each period.

Much of the seminal literature on growth optimising investment strategies concentrates on the case of one risky and one risk-free asset. Furthermore, the literature has grown out of the particular case in which the risky asset is distributed according to a simple binomial risk. Concretely, assume (for now) that *in each and every period* the risky investment pays off at a per dollar invested with probability p (the "attractive" outcome) and at b per dollar invested with probability $1-p$ (the "bad" outcome). Assume that the risk-free alternative pays off at r per dollar invested (with probability 1). These asset returns are constrained such that $a > r > b$ and $pa + (1-p)b > r$, so that there is some risk in the risky investment relative to holding the risk-free investment, and there is also some incentive to participate in the risky investment. The investor starts with initial wealth $w_0 = 1$, and invests a fraction λ_i of current wealth in the risky investment in each period i. Furthermore, assume that the investor's objective is to elevate the expected growth of wealth as much as possible. As we have just seen, this is equivalent to maximising the expectation of the log of periodic wealth, i.e. the investor should act as if she were an expected utility maximiser with log utility function.

Second, because of the constant relative risk aversion of the log utility function, the fraction of wealth dedicated to the investment in each period is independent of the amount of current wealth, and can only depend on the characteristics of the investment payoffs. However, our assumption for now is that the investment payoffs and probabilities are identical from one period to the next. This directly tells us that the fraction of current wealth that will be dedicated to the investment in each period must necessarily be a constant, that is, $\lambda_i = \lambda$ for all periods i. The optimal investment strategy for this situation, assuming the investor has the objective of maximising the long-term growth of total wealth, is (see Appendix 1 for details):

$$\lambda^* = \frac{(1+r)(pa + (1-p)b - r)}{(a-r)\,(r-b)} \tag{1}$$

This famous formula is known as the *"Kelly captial growth criterion"*, named after John Kelly, who was an information theory scientist with Bell Labs in the 1950s, and who published a very influential paper related to this topic in 1956.[1]

For example, if the risky investment pays at a rate of interest of $a = 0.25$ with probability 0.4, and makes a

[1] Kelly was not a finance scholar, but rather was interested in what is known as "information theory". His paper, which was inspired by the work of Claude Shannon (1948), is about how one should value information flows over time, but the main result of the paper has very clear and obvious relevance for optimal investment strategies in dynamic finance. The Kelly criterion was hotly disputed in the academic literature, with one of its main opponents being no less than Nobel Laureate Paul Samuelson, who argued that it is unreasonable to assume that every investor's risk preferences will coincide with log utility. That is, someone more risk averse than log utility would not actually be interested in maximising the long term growth of their fortune, since that can come at the cost of excessive risk (see, for example, the rather snide final paper on this topic by Samuelson, 1979).

rate of interest of $b = 0.01$ with probability 0.6 (thus, there is an expected gain in each period of 10.6%), in a world in which the risk-free rate is $r = .1$, then the optimal investment is

$$\lambda^* = \frac{(1 + 0.1)(0.4 \times 0.25 + 0.6 \times 0.01 - 0.1)}{(0.25 - 0.1)\,(0.1 - 0.01)} = 0.488$$

i.e. just under 49% of the available wealth should be dedicated to the risky asset in each period.

As it happens, even though (1) is the "Kelly Capital Growth Criterion", the original author, John Kelly and almost all of the follow-on literature, only resolved a particular special case. Concretely, the common assumptions are $r = 0$ and the downside of the bet is that one loses what was wagered ($b = -1$). The corresponding Kelly criterion optimal investment is $p - \frac{1-p}{a}$.

Go back to our more general formula, equation (1). In all that follows, define $pa + (1 - p)b \equiv \widehat{\mu}$, so that

$$\lambda^*(a, b, r) = \frac{(1 + r)(\widehat{\mu} - r)}{(a - r)\,(r - b)}$$

Furthermore, attention is restricted only to cases in which $a > r > b$, $\widehat{\mu} > r$ and $0 < p < 1$.

It is easy to see that $\lambda^* = 0$ (i.e. don't participate at all in the risky asset) if $\widehat{\mu} - r = 0$, that is, if the expected value of the risky return is only equal to the risk-free return - the expected payoff from the investment is no better than holding a fully risk-free portfolio. This, and even negative values of λ^* (the risky asset is sold short in order to purchase more of the risk-free asset) are ruled out by our assumptions on the parameter values, and so they will be ignored here. But as soon as the expected payoff exceeds r, this investor always desires some (possibly a very small) amount of wealth in the risky investment. This is, of course, a common result in all models of one risky asset and one risk-free asset.

Second, we get $\lambda^* = 1$, that is, invest all of the available fund in the risky asset, if the risk-free rate is the r' that satisfies

$$(1 + r')(\widehat{\mu} - r') = (a - r')\,(r' - b)$$

Straight-forward steps show that this equation is

$$r' = \frac{ab + \widehat{\mu}}{1 + a + b - \widehat{\mu}}$$

For this to be valid with a strictly positive value of r', we require $1 + a + b - \widehat{\mu} > 0$. While we cannot guarantee this will always hold, it will only *not* hold if b is a very small (and negative) number. Concretely, if $b > -1$, then $1 + a + b - \widehat{\mu} > 0$ holds for sure. Notice that in this game the only chance of bankruptcy is if the investor goes all in on the risky asset, and if $b \leq -1$. Take the case of $b = -1$, which gives $1 + a + b - \widehat{\mu} = a - \widehat{\mu} > 0$.

This gives

$$r' = \frac{-a+\widehat{\mu}}{a-\widehat{\mu}} = -1$$

An investor facing a risk with potential bankruptcy will only go all in on that risk if the risk-free rate delivers bankruptcy for sure. This is, of course, an absurd situation. For any other $b > -1$, we get $1+a+b-\widehat{\mu} > a-\widehat{\mu} > 0$. Further, notice that when $b > -1$, we get $r > b$ if

$$\frac{ab+\widehat{\mu}}{1+a+b-\widehat{\mu}} > b$$

Again, straight-forward steps show that this is just $\widehat{\mu} > b$, which is satisfied by our initial assumptions. This tells us that the optimal Kelly investment provides a guard against bankruptcy – the only cases in which the investor goes all in on the risk are cases in which the risk cannot deliver bankruptcy.

Next, consider the first-derivatives of λ^* with respect to the variables comprising it. First, we have

$$\begin{aligned}\frac{\partial\lambda^*}{\partial a} &= \frac{\frac{\partial\widehat{\mu}}{\partial a}(1+r)(a-r)(r-b)-(1+r)(\widehat{\mu}-r)\,(r-b)}{((a-r)\,(r-b))^2} \\ &= \frac{p(1+r)(a-r)(r-b)-(1+r)(\widehat{\mu}-r)\,(r-b)}{((a-r)\,(r-b))^2}\end{aligned}$$

The sign of this is the same as the sign of the numerator,

$$\frac{\partial\lambda^*}{\partial a} \gtreqless 0 \quad \text{as} \quad p(1+r)(a-r)(r-b) \gtreqless (1+r)(\widehat{\mu}-r)\,(r-b)$$

Cancelling common terms and simplifying, we get:

$$\frac{\partial\lambda^*}{\partial a} \gtreqless 0 \quad \text{as} \quad r \gtreqless b$$

Therefore, (as we should expect) it happens that $\frac{\partial\lambda^*}{\partial a} > 0$, i.e. more is invested in the risky asset the higher is the best payoff (all else equal).

Similarly, we should expect that the higher is the bad payoff from the risky asset, all else equal, more will be invested in the risk. Indeed, we have

$$\begin{aligned}\frac{\partial\lambda^*}{\partial b} &= \frac{\frac{\partial\widehat{\mu}}{\partial b}(1+r)(a-r)(r-b)+(1+r)(\widehat{\mu}-r)\,(a-r)}{((a-r)\,(r-b))^2} \\ &= \frac{(1-p)(1+r)(a-r)(r-b)+(1+r)(\widehat{\mu}-r)\,(a-r)}{((a-r)\,(r-b))^2}\end{aligned}$$

Notice that both the numerator and the denominator are positive. Therefore, it holds that $\frac{\partial\lambda^*}{\partial b} > 0$.

Next, let's check the effect of an increase in p;

$$\frac{\partial\lambda^*}{\partial p} = \frac{\frac{\partial\widehat{\mu}}{\partial p}(1+r)}{(a-r)(r-b)} = \frac{(a-b)(1+r)}{(a-r)(r-b)} > 0$$

So, as is natural, an increase in p, which again makes the risky asset more attractive, leads the investor to invest more heavily in the risk.

Finally, consider the effect of an increase in the risk-free rate r. As it happens, signing the derivative $\frac{\partial\lambda^*}{\partial r}$ itself is complex. It is far easier to consider the graph of the function in question. We know the following facts:

1. There are two asymptotes for the graph of $\lambda^*(r)$, a left-side one at $r = b$ and a right-side one at $r = a$.

2. $\lambda^* = 0$ at $r = \widehat{\mu}$. Furthermore, this is the only point at which $\lambda^* = 0$.

The graph of λ^* as a function of r is the following (drawn here for $a = 0.1$, $b = -0.15$ and $p_b = 0.2$), so $\widehat{\mu} = 0.05$ and $\widehat{\sigma} = 0.1$ (i.e. $\widehat{\sigma}^2 = 0.01$):

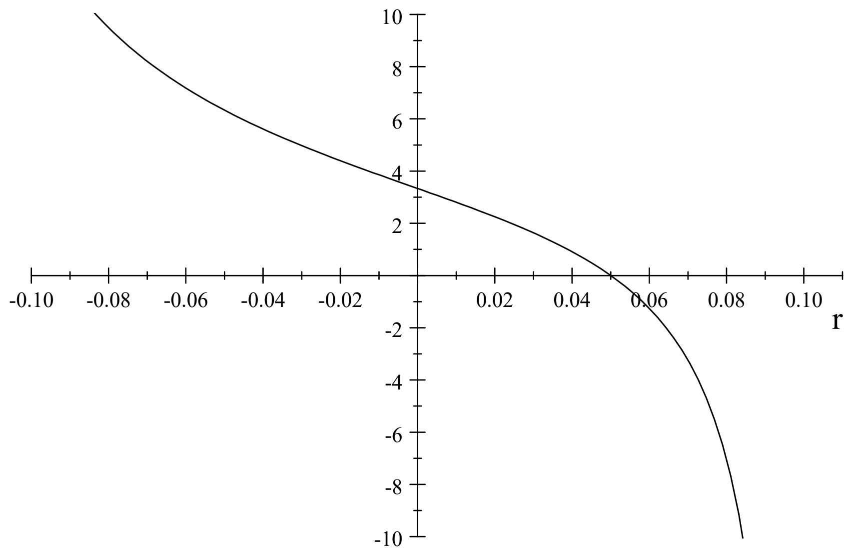

Figure 1: The graph of λ^* as a function of r

In Figure 1, we can indeed see that, as we should expect, λ^* falls as r rises, that is $\frac{\partial\lambda^*}{\partial r} < 0$. However, nothing much at all changes in regards the shape of the function with other choices of the parameters, so long as we keep to the restrictions $a > r > b$, $\widehat{\mu} > r$ and $0 < p < 1$.

3 Discrete time Kelly investing with mean-standard deviation information

While Kelly only considered risky investments that are described by two possible outcomes, it is natural to consider more general risks. When the risky asset gives a return that is stochastic, defined by a known probability density function, nothing in particular changes in regards the main message of the Kelly Criterion – the optimal strategy for an investor with the objective of maximising the growth of their fund is to use the optimal investment strategy of an expected utility maximiser with log utility. However, in keeping with standard practice in financial decision making, it is useful to attempt to describe the risk involved in terms of its moments, and in particular in terms of its first two moments – expected value and standard deviation – rather than working with the general probability density function.

In order to do that, of course, some sort of approximation is required in order to reduce the description of the risky asset down to only its first two moments. To that end, the best known approximation is due to Edward O. Thorp (2006), so I shall concentrate on that here.

3.1 Thorp's approximation

Thorp's approximation is set out in Thorp (2006).[2] Thorp's idea is based on describing the general risk in terms of a symmetric binomial lottery. Concretely, Thorp assumed that the risky return, $\widetilde{x}$ is described by

$$\widetilde{x} = \left[\widehat{\mu} + \widehat{\sigma}, \widehat{\mu} - \widehat{\sigma}; \frac{1}{2}, \frac{1}{2}\right]$$

That is, the risk gives a good outcome of $a = \widehat{\mu} + \widehat{\sigma}$ and a bad outcome of $b = \widehat{\mu} - \widehat{\sigma}$, each with equal probability, where $\widehat{\mu}$ is the mean outcome and $\widehat{\sigma}$ is the standard deviation of the outcome. Given this assumption, the risky asset is now described in perfect binomial terms, so we can directly substitute into the general formula for the Kelly Criterion, to get an equation that only requires information on means and standard deviations:[3]

$$\lambda_B^*(\widehat{\mu}, \widehat{\sigma}, r) = \frac{(1+r)(\widehat{\mu} - r)}{(\widehat{\mu} + \widehat{\sigma} - r)\,(r - (\widehat{\mu} - \widehat{\sigma}))} = \frac{(1+r)(\widehat{\mu} - r)}{\widehat{\sigma}^2 - (\widehat{\mu} - r)^2} \tag{2}$$

The same assumptions as previously apply, above all $\widehat{\mu} + \widehat{\sigma} > r > \widehat{\mu} - \widehat{\sigma}$, which tell us that the denominator of this expression is positive.

Again, we can directly see that;

[2] See also Merton (1969), and Rotando and Thorp (1992).

[3] Here, the sub-index B indicates that the binomial assumption has been used. I make no claim of originality for this formula, which is very easily derived. Even so, it does not appear at all in any of the papers in the extensive work on the existing literature up to 2011 on the Kelly Criterion (MacLean et al., 2011).

1. $\lambda_B^* = 0$ if and only if $\widehat{\mu} = r$. Otherwise $\lambda_B^* > 0$.

2. $\lambda_B^* = 1$ if $(1+r)(\widehat{\mu} - r) = \widehat{\sigma}^2 - (\widehat{\mu} - r)^2$. This reduces to $r = \widehat{\mu} - \frac{\widehat{\sigma}^2}{1+\widehat{\mu}}$.

3. $\frac{\partial \lambda_B^*}{\partial \widehat{\mu}} > 0$, so an increase in the mean of the risky asset raises the optimal investment in risk.

4. $\frac{\partial \lambda_B^*}{\partial \widehat{\sigma}} < 0$, so an increase in the standard deviation of the risky asset reduces the optimal investment in risk.

5. $\frac{\partial \lambda_B^*}{\partial r} < 0$, so an increase in the risk-free rate reduces the optimal investment in risk.[4]

While Thorp doesn't give any specific argument to support his binomial methodology, it is clear that the assumption is based on the following. Standard portfolio theory suggests that given any risky asset with coordinates $(\widehat{\mu}, \widehat{\sigma})$, an investor will be indifferent between that asset and any other with the same mean-standard deviation coordinates. That is, any 2-dimensional risk with the same coordinates works as a perfect substitute for the risky asset in question. Therefore, take a 2-dimensional asset described by payoffs and probabilities that satisfy $[a, b; p, 1-p]$, with $a > b$. Then, solve the following two simultaneous equations in a and b

$$\begin{aligned} pa + (1-p)b &= \widehat{\mu} \\ p(1-p)(a-b)^2 &= \widehat{\sigma}^2 \end{aligned}$$

Straight-forward calculations reveal that that the solution is

$$\begin{aligned} a &= \widehat{\mu} + \widehat{\sigma}\sqrt{\frac{1-p}{p}} \\ b &= \widehat{\mu} - \widehat{\sigma}\sqrt{\frac{p}{1-p}} \end{aligned}$$

Since any particular p can be used, Thorp takes the simplest possible value, namely $p = \frac{1}{2}$. The following graph shows (2) drawn as a function of the risk-free rate r, using $\widehat{u} = 0.05$ and $\widehat{\sigma}^2 = 0.01$;

[4]While the other two derivatives are relatively simple to check, just as in the earlier model, it is very complicated to show that the derivative in r is in fact negative. However, once again a simple graphical check does indeed confirm that to be the case (see Figure 2 below).

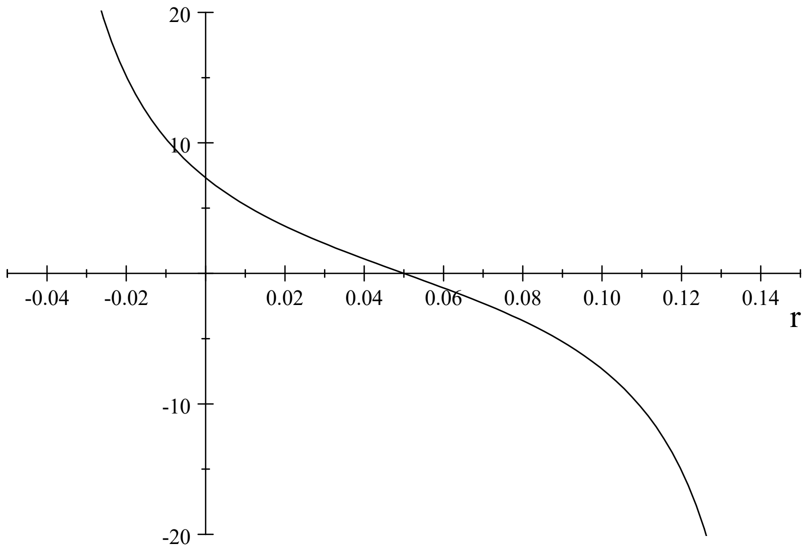

Figure 2: λ_B^* as a function of r.

Two caviats for Thorp's approximation are in order. First, regardless of what the underlying risk actually is, Thorp's methodology assumes it is a symmetric binomial. The approximation then would always work better if the actual underlying risk were a symmetric binomial (even if the two payoffs are not exactly those Thorp assumes). In particular, the more heavily skewed is the true risk, the worse will be the approximation. Second, we should bear in mind that Borch (1969) has shown that even though two assets share the same mean-standard deviation coordinates, with all other statistical moments equal to 0, they are actually not perfect substitutes in any preference functional that is increasing in mean and decreasing in standard deviation. This is known as the "Borch paradox".[5]

In Thorp (2006), the formula here shown as (2) does not appear, although it is certainly implied by Thorp's analysis. In fact, on top of the assumption that the risky investment can be described by a symmetric binomial approximation, Thorp further refined his method using a logarithmic approximation and an assumption of continuous time to arrive at his final suggested approximation, which is[6]

$$T(r) = \frac{\widehat{\mu} - r}{\widehat{\sigma}^2} = \frac{\widehat{\mu}}{\widehat{\sigma}^2} - \frac{1}{\widehat{\sigma}^2} r \tag{3}$$

Notice that this alternative is very closely related to the Sharpe ratio (it is the Sharpe ratio divided by standard deviation). In all that follows, I shall refer to (3) as "Thorp's linear approximation".

[5] Borch insisted upon this point in a couple of follow-up articles (Borch 1973, 1974). It was a hotly debated issue for a while, but more recently it seems to have simply been swept under the carpet and largely ignored. For a relatively recent discussion on the issue, see Johnstone and Lindley (2013).

[6] Others have also come to the same result under the same assumption of continuous time. Perhaps most notably, see Merton (1969).

It is easy to see the following facts about Thorp's linear approximation:

1. $T(r) = 0$ if and only if $\widehat{\mu} = r$. For all $\widehat{\mu} > r$, we get $T(r) > 0$, so some investment in risk is always optimal whenever the risk pays of at a higher expectation than the risk-free option.

2. $T(r)$ increases with $\widehat{\mu}$, decreases with r, and decreases with $\widehat{\sigma}$.

3. $T(r)$ is a linear function of r.

4. For all admissible parameter values, $T(r) < \lambda_B^*$, that is, Thorp's linear approximation routinely underestimates λ_B^*.

Point 4 is the only one that requires some calculation. It is true if

$$\frac{\widehat{\mu} - r}{\widehat{\sigma}^2} < \frac{(1+r)(\widehat{\mu} - r)}{\widehat{\sigma}^2 - (\widehat{\mu} - r)^2}$$

To avoid a tautology, assume $\widehat{\mu} > r$. Simple operations then lead to

$$\widehat{\sigma}^2 - (\widehat{\mu} - r)^2 < \widehat{\sigma}^2(1+r)$$

or

$$-(\widehat{\mu} - r)^2 < \widehat{\sigma}^2 r$$

which is trivially true since the left-hand side is negative and the right-hand side is positive.

Points 1 and 3 here are interesting, and they make it tempting to think that $T(r)$ is the first-order Taylor's approximation to λ_B^*, around the point $r = \widehat{\mu}$. However, straight-forward calculations reveal that in fact the first-order Taylor's approximation at $r = \widehat{\mu}$ is

$$\begin{aligned} R_B(r) &= -\frac{(1+\widehat{\mu})(r - \widehat{\mu})}{\widehat{\sigma}^2} \\ &= \frac{(1+\widehat{\mu})(\widehat{\mu} - r)}{\widehat{\sigma}^2} \\ &= \frac{\widehat{\mu}(1+\widehat{\mu})}{\widehat{\sigma}^2} - \frac{(1+\widehat{\mu})}{\widehat{\sigma}^2} r \end{aligned}$$

Notice that this can also be expressed as

$$R_B(r) = (1+\widehat{\mu}) \times T(r)$$

That is, the first-order Taylor's approximation to λ_B^* is equal to Thorp's linear approximation, $T(r)$, multiplied by a correcting factor $1+\widehat{\mu}$.[7] Since $\widehat{\mu} > 0$, it will always happen that $T(r) < R_B(r)$. Furthermore, we have the followng result:

Proposition 1 *If $R_B(r) < 1$ then $R_B(r) > \lambda_B^*$, if $R_B(r) > 1$ then $R_B(r) < \lambda_B^*$, and if $R_B(r) = 1$ then $R_B(r) = \lambda_B^*$.*

Proof. Consider the relationship between λ_B^* and $R_B(r)$:

$$\begin{aligned}
\frac{(\widehat{\mu}-r)(1+\widehat{\mu})}{\widehat{\sigma}^2} &\lesseqgtr \frac{(1+r)(\widehat{\mu}-r)}{\widehat{\sigma}^2-(\widehat{\mu}-r)^2} \\
(1+\widehat{\mu})\left(\widehat{\sigma}^2-(\widehat{\mu}-r)^2\right) &\lesseqgtr \widehat{\sigma}^2(1+r) \\
(1+\widehat{\mu})\,\widehat{\sigma}^2-(1+\widehat{\mu})(\widehat{\mu}-r)^2 &\lesseqgtr \widehat{\sigma}^2(1+r) \\
(\widehat{\mu}-r)\,\widehat{\sigma}^2 &\lesseqgtr (1+\widehat{\mu})(\widehat{\mu}-r)^2 \\
\widehat{\sigma}^2 &\lesseqgtr (1+\widehat{\mu})(\widehat{\mu}-r) \\
1 &\lesseqgtr \frac{(1+\widehat{\mu})(\widehat{\mu}-r)}{\widehat{\sigma}^2} = R_B(r)
\end{aligned}$$

■

The fact that $R_B(r) = \lambda_B^*$ at $\lambda_B^* = 1$, and they are also both equal to 0 at $r = \widehat{\mu}$, mean that $R_B(r)$ is simply the straight line joining the point $r = \mu$, with the point at which $\lambda_B^* = 1$.

Figure 3 shows three curves, namely $T(r)$, $R_B(r)$, and λ_B^*, over a small but highly relevant range of values of the risk-free return, drawn for the same parameter values as Figure 2. In the graph, for values of r between about 0.035 and 0.05 the second two curves ($R_B(r)$ and λ_B^*) are so close together that they appear as if they were one.

[7] Notice that, since initial wealth has been normalized to 1, the correcting factor $1+\widehat{\mu}$ is just the expected value of initial wealth at the end of the decision period, should all of the initial wealth be invested in the risky asset. On the other hand, $T(r)$ is implicitly pre-multiplied by initial wealth (normalized to 1).

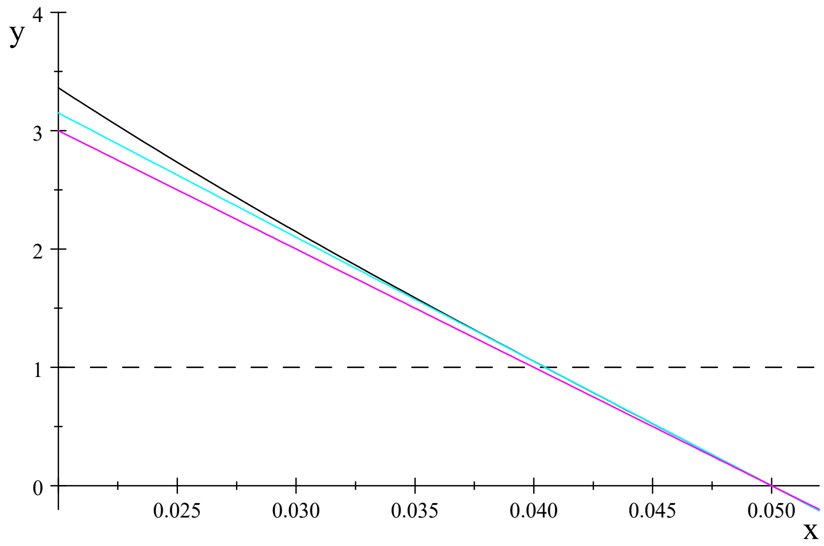

Figure 3: Comparison of $R_B(r)$ (blue curve), $T(r)$ (pink curve) and λ_B^* (black curve, obscured behind the blue one).

Of particular interest is the point at which the different strategies ask the investor to go all-in on the risk. The exact value of r for which both λ_B^* and $R_B(r)$ are equal to 1 is

$$\begin{aligned}
\frac{\widehat{\mu}(1+\widehat{\mu})}{\widehat{\sigma}^2} - \frac{(1+\widehat{\mu})}{\widehat{\sigma}^2} r &= 1 \\
\frac{\widehat{\mu}(1+\widehat{\mu})}{\widehat{\sigma}^2} - 1 &= \frac{(1+\widehat{\mu})}{\widehat{\sigma}^2} r \\
\left(\frac{\widehat{\mu}(1+\widehat{\mu})}{\widehat{\sigma}^2} - 1\right) \frac{\widehat{\sigma}^2}{(1+\widehat{\mu})} &= r \\
r &= \widehat{\mu} - \frac{\widehat{\sigma}^2}{(1+\widehat{\mu})}
\end{aligned}$$

On the other hand, $T(r) = 1$ gives

$$\begin{aligned}
\frac{\widehat{\mu} - r}{\widehat{\sigma}^2} &= 1 \\
r &= \widehat{\mu} - \widehat{\sigma}^2
\end{aligned}$$

With the example above, both λ_B^* and $R_B(r)$ are equal to 1 at $r = 0.040476$, while $T(r) = 1$ at $r = 0.04$. Thus, with the parameter values used in Figure 3, $T(r)$ instructs the investor to wait until the risk-free rate is 4% before going all-in on the risky asset, whereas both λ_B^* and $R_B(r)$ ask that the investor goes all-in when the risk-free rate falls to the slightly higher value 4.05%.

The facts that (i) both $T(r)$ and $R_B(r)$ are linear, (ii) $T(r) < R_B(r)\ \forall r$ (ii) $T(r) < \lambda_B^*\ \forall r$ (iii) $R_B(r) < \lambda_B^*$ $\forall r : \lambda_B^* > 1$, together imply that $R_B(r)$ approximates λ_B^* more closely than does $T(r)$ for all r such that $\lambda_B^* > 1$:[8]

Proposition 2 $\lambda_B^* - R_B(r) < \lambda_B^* - T(r)$ *for all* r *such that* $\lambda_B^* > 1$.

What about the range of r values for which $1 > \lambda_B^* > 0$? Figure 3 above seems to suggest that $R_B(r)$ will still be a closer approximation to λ_B^* than is $T(r)$, although on that zone $R_B(r)$ ever-so-slightly overestimates λ_B^* while $T(r)$ underestimates λ_B^*. As it happens, the following result holds:

Proposition 3 *For any risky asset that satisfies* $(2+\widehat{\mu})\,\widehat{\mu} \geq \widehat{\sigma}^2$, $R_B(r)$ *is a closer approximation to* λ_B^* *than is* $T(r)$ *for all values of* r.

Proof. For all values of r such that $\lambda_B^* > 1$, the previous proposition holds, and $R_B(r)$ approximates λ_B^* better than does $T(r)$. So we only need to consider the zone of points for which $1 > \lambda_B^* > 0$. Since on that zone, we know that the error commited by the first-order Taylor's approximation is $R_B(r) - \lambda_B^* > 0$ and that the error commited by Thorp's approimation is $\lambda_B^* - T(r) > 0$, the former is smaller than the latter if $R_B(r) - \lambda_B^* < \lambda_B^* - T(r)$, which is the average of the two approximations would need to be less than the function being approximated:

$$\frac{R_B(r) + T(r)}{2} < \lambda_B^*$$

Sucessive, but tedious, operations reduse this to:

$$-\left(2+\widehat{\mu}\right) r^2 + 2\left(\left(2+\widehat{\mu}\right)\widehat{\mu} - \widehat{\sigma}^2\right) r + \left(\widehat{\mu}\widehat{\sigma}^2 - \left(2+\widehat{\mu}\right)\widehat{\mu}^2\right) < 0$$

This is a standard concave second-order polinomial in r. Thus, it is everywhere less than its value at the turning point. The turning point can easily be seen to satisfy

$$-2\left(2+\widehat{\mu}\right) r_0 + 2\left(\left(2+\widehat{\mu}\right)\widehat{\mu} - \widehat{\sigma}^2\right) = 0$$

So it is at

$$\begin{aligned} r_0 &= \frac{\left(2+\widehat{\mu}\right)\widehat{\mu} - \widehat{\sigma}^2}{2+\widehat{\mu}} \\ &= \widehat{\mu} - \frac{\widehat{\sigma}^2}{2+\widehat{\mu}} \end{aligned}$$

[8] A value of λ that is greater than 1 instructs the investor to borrow at the risk-free rate and invest all funds in the risky asset.

Substitute r_0 back into the polinomial:

$$
\begin{aligned}
& -\left(2+\widehat{\mu}\right)\left(\frac{\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2}{2+\widehat{\mu}}\right)^2 + 2\left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)\left(\frac{\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2}{2+\widehat{\mu}}\right) + \widehat{\mu}\left(\widehat{\sigma}^2-\left(2+\widehat{\mu}\right)\widehat{\mu}\right) \\
= & \frac{\left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)^2}{2+\widehat{\mu}} + \widehat{\mu}\left(\widehat{\sigma}^2-\left(2+\widehat{\mu}\right)\widehat{\mu}\right) \\
= & \frac{\left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)^2}{2+\widehat{\mu}} - \widehat{\mu}\left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right) \\
= & \left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)\left(\frac{\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2}{2+\widehat{\mu}} - \widehat{\mu}\right) \\
= & \left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)\left(\widehat{\mu} - \frac{\widehat{\sigma}^2}{2+\widehat{\mu}} - \widehat{\mu}\right) \\
= & \left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)\left(-\frac{\widehat{\sigma}^2}{2+\widehat{\mu}}\right) \\
= & -\left(\left(2+\widehat{\mu}\right)\widehat{\mu}-\widehat{\sigma}^2\right)\left(\frac{\widehat{\sigma}^2}{2+\widehat{\mu}}\right)
\end{aligned}
$$

Notice that this is no greater than 0 if

$$\left(2+\widehat{\mu}\right)\widehat{\mu} \geq \widehat{\sigma}^2$$

■

It is rather easy that the condition $\left(2+\widehat{\mu}\right)\widehat{\mu} \geq \widehat{\sigma}^2$ is fulfilled for most relevant risky assets. As a very quick and easy check, since $\widehat{\mu} > 0$, the condition is surely fulfilled for any risky asset such that $2\widehat{\mu} \geq \widehat{\sigma}^2$.[9] Thus, for a very large family of risky assets, the first-order Taylor's approximation to λ_B^* performs uniformly better than does Thorp's approximation.

3.2 Markowitzian approximation

I now move onto a second approximation methodology, that is not based on the binomial assumption used by Thorp, and that does not require an assumption of continuous time. Since a Kelly investor acts exactly as if they were an expected utility maximiser with log utility, it is relatively simple to see how such an investor would operate directly in the mean-standard deviation (MSD) environment, as pioneered by Markowitz (1952).[10] All we need to do is to employ the standard second-order approximation to log utility preferences,[11] in order to search for a tangency on the investment opportunities line joining the risk-free point with the point representing

[9] Even easier, the condition is fulfilled if variance is smaller than the mean. In fact, since mean and variance are measured in different units, their comparison is somewhat meaningless in theory. But, they are also both numbers, and can always be compared numerically. For example, in the binomial distribution variance is always smaller than the mean. In the Poisson distribution, the two are equal. For the exponential distribution, variance is smaller than mean if standard deviation is less than 1.

[10] See also Tobin (1965).

[11] As ever, when this is done there is again a certain cost in precision, since the MSD environment can only admit an approximation to actual preferences.

the risky asset. This gives rise to an optimum that will look somewhat like that shown in Figure 4:

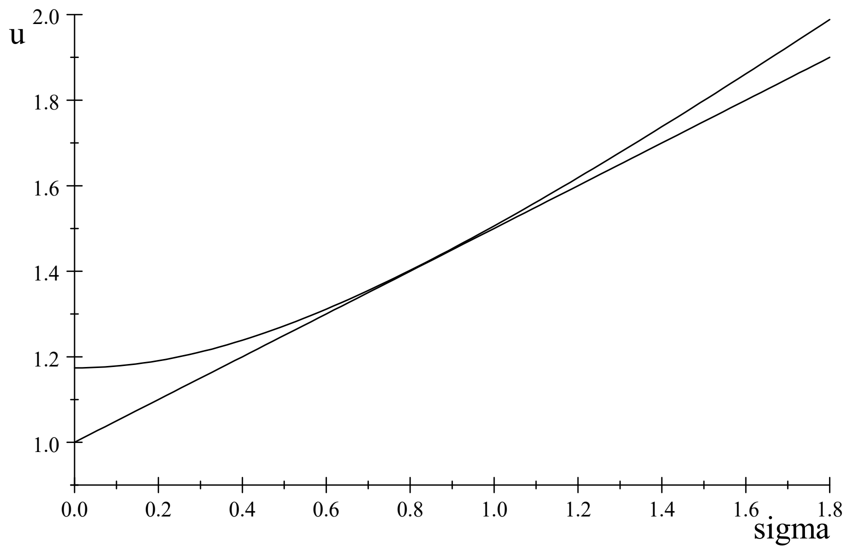

Figure 4: An optimal portfolio in Markowitz space

In order to distinguish this method for a Kelly investor from the one considered above, I will refer to this as the "Markowitzian" approximation, since it is grounded on the graphical environment made famous by Harry Markowitz.[12]

In general, the second-order approximation to expected utility is

$$v(\mu_w, \sigma_w) = \mu_w - \frac{\sigma_w^2}{2} R^a(\mu_w)$$

where μ_w is the mean of final wealth, σ_w is the standard deviation of final wealth, and $R^a(w)$ is the absolute risk aversion function. With $u(w) = \ln(w)$, we get $R^a(w) = \frac{1}{w}$. Substituting this into the second-order approximation to preferences, we immediately get

$$v(\mu_w, \sigma_w) = \mu_w - \frac{\sigma_w^2}{2\mu_w}$$

which is an objective function that is directly representable in MSD space (μ_w, σ_w). The corresponding marginal

[12] While the approximation is built from Markowitz's methodology, Markowitz himself did not himself suggest the ensuring approximation to the optimal Kelly investment.

rate of substitution in the space (μ_w, σ_w) is (from the implicit function theorem);

$$\begin{aligned}
\frac{d\mu_w}{d\sigma_w} &= -\frac{\left(\frac{\partial v}{\partial \sigma_w}\right)}{\left(\frac{\partial v}{\partial \mu_w}\right)} \\
&= -\frac{\left(-\frac{\sigma_w}{\mu_w}\right)}{\left(1+\frac{\sigma_w^2}{2\mu_w^2}\right)} \\
&= \frac{\left(\frac{\sigma_w}{\mu_w}\right)}{\left(\frac{2\mu_w^2+\sigma_w^2}{2\mu_w^2}\right)} \\
&= \frac{\sigma_w}{\mu_w} \times \frac{2\mu_w^2}{2\mu_w^2+\sigma_w^2} \\
&= \frac{2\sigma_w \mu_w}{2\mu_w^2+\sigma_w^2}
\end{aligned}$$

Of course, this is positive if both μ_w and σ_w are positive. It is also exactly 0 at any risk-free point ($\sigma_w = 0$).

Now, given a risk-free investment with per-unit return r, and given a single risky investment opportunity with mean return $\widehat{\mu}$ and standard deviation $\widehat{\sigma}$, and given wealth divided over the two according to a share λ held in the risky asset, we can calculate the expressions for μ_w and σ_w. Concretely, continuing with the normalisation of initial wealth to 1, we get

$$\begin{aligned}
\mu_w &= E\left(\lambda\left(1+\widetilde{x}\right)+\left(1-\lambda\right)\left(1+r\right)\right) \\
&= E\left(\left(1+r\right)+\lambda\left(\widetilde{x}-r\right)\right) \\
&= \left(1+r\right)+\lambda\left(\widehat{\mu}-r\right)
\end{aligned}$$

and

$$\begin{aligned}
\sigma_w^2 &= E((1+r)+\lambda\left(\widetilde{x}-r\right)-(1+r)+\lambda\left(\widehat{\mu}-r\right))^2 \\
&= E(\lambda\left(\widetilde{x}-r\right)-\lambda\left(\widehat{\mu}-r\right))^2 \\
&= \lambda^2 E(\widetilde{x}-\widehat{\mu})^2 \\
&= \lambda^2 \widehat{\sigma}^2
\end{aligned}$$

Next, the investment opportunities line is given by the linear expression

$$\mu_w = (1+r) + \frac{\widehat{\mu}-r}{\widehat{\sigma}}\sigma_w$$

The slope of this line is $\frac{\widehat{\mu}-r}{\widehat{\sigma}}$, which is commonly known as the Sharpe ratio. It is convenient to simply define

$$\frac{\widehat{\mu}-r}{\widehat{\sigma}} \equiv S$$

The Kelly investment choice is the point on this investment opportunities line where the marginal rate of substitution is equal to the slope of the investment opportunities line. We thus have two equations in two unknowns;

$$\frac{2\sigma_w^* \mu_w^*}{2\mu_w^{*2} + \sigma_w^{*2}} = S \quad \text{and} \quad \mu_w^* = (1+r) + S\sigma_w^*$$

Write the second of these equations as

$$\sigma_w^* = \frac{\mu_w^* - (1+r)}{S}$$

and directly substitute this into the first equation, to get (with successive reductions)

$$\begin{aligned}
\frac{2\left(\frac{\mu_1^*-(1+r)}{S}\right)\mu_w^*}{2\mu_w^{*2} + \left(\frac{\mu_w^*-(1+r)}{S}\right)^2} &= S \\
2\left(\frac{\mu_w^* - (1+r)}{S}\right)\mu_w^* &= S\left(2\mu_w^{*2} + \left(\frac{\mu_w^* - (1+r)}{S}\right)^2\right) \\
2\left(\frac{\mu_w^* - (1+r)}{S}\right)\mu_w^* &= 2S\mu_w^{*2} + S\left(\frac{\mu_w^* - (1+r)}{S}\right)^2 \\
2\left(\mu_w^* - (1+r)\right)\mu_w^* &= 2S^2\mu_w^{*2} + \left(\mu_w^* - (1+r)\right)^2 \\
2\mu_w^{*2} - 2(1+r)\mu_w^* &= 2S^2\mu_w^{*2} + \mu_w^{*2} + (1+r)^2 - 2(1+r)\mu_w^* \\
2\mu_w^{*2} &= 2S^2\mu_w^{*2} + \mu_w^{*2} + (1+r)^2 \\
\mu_w^{*2}\left(2 - 2S^2 - 1\right) &= (1+r)^2 \\
\mu_w^{*2}\left(1 - 2S^2\right) &= (1+r)^2
\end{aligned}$$

At this point, we need to notice that this is only consistent with a solution if $1 - 2S^2 > 0$, and so we will assume this to hold (see Appendix 2 for a detailed consideration of this assumption). Then, we end up with the following

$$\begin{aligned}
\mu_w^{*2} &= \frac{(1+r)^2}{1-2S^2} \\
\mu_w^* &= \frac{1+r}{\sqrt{1-2S^2}}
\end{aligned}$$

Notice that unless $S = 0$, which is the same as $\widehat{\mu} = r$, it must happen that $\sqrt{1-2S^2} < 1$, and so $\mu_w^* > 1 + r$. This reminds us that when the risky asset pays off at an expected interest rate that is greater than the risk-free rate, the investor will always like to include some of that risk in her portfolio. Alternatively, if the risky asset pays of at an expected rate that equals the risk-free rate ($S = 0$), then the optimal investment is to exclude the risky asset (giving $\mu_w^* = 1 + r$). Finally, using the above equation for μ_w^*, we can calculate the corresponding standard deviation coordinate;

$$\begin{aligned}
\sigma_w^* &= \frac{\mu_w^* - (1+r)}{S} \\
&= \frac{\frac{(1+r)}{\sqrt{1-2S^2}} - (1+r)}{S} \\
&= \frac{(1+r)}{S\sqrt{1-2S^2}} - \frac{(1+r)}{S} \\
&= \frac{(1+r)}{S}\left(\frac{1}{\sqrt{1-2S^2}} - 1\right) \\
&= \frac{(1+r)}{S}\left(\frac{1-\sqrt{1-2S^2}}{\sqrt{1-2S^2}}\right)
\end{aligned}$$

The optimal investment fraction, λ^*, can be directly calculated by setting the equation for optimal mean of final wealth equal to the equation for mean of final wealth itself (where on the last line, the optimal investment in risk is indexed with M (for "Markowitz", since this approximation is found using the standard Markowtiz portfolio environment);

$$\begin{aligned}
\frac{1+r}{\sqrt{1-2S^2}} &= (1+r) + \lambda^*\left(\widehat{\mu} - r\right) \\
\lambda^*\left(\widehat{\mu} - r\right) &= \frac{1+r}{\sqrt{1-2S^2}} - (1+r) \\
\lambda^*\left(\widehat{\mu} - r\right) &= (1+r)\left(\frac{1}{\sqrt{1-2S^2}} - 1\right) \\
\lambda^*\left(\widehat{\mu} - r\right) &= (1+r)\left(\frac{1-\sqrt{1-2S^2}}{\sqrt{1-2S^2}}\right) \\
\lambda_M^* &= \left(\frac{1+r}{\widehat{\mu} - r}\right)\left(\frac{1-\sqrt{1-2S^2}}{\sqrt{1-2S^2}}\right) \qquad ((4))
\end{aligned}$$

The graph of this optimal investment is shown below in Figure 5, once again using the same parameter values ($\widehat{\mu} = 0.05$, $\widehat{\sigma}^2 = 0.01$) as in our graphs of the optimal investment strategy under Thorp's binomial assumption, λ_B^*.

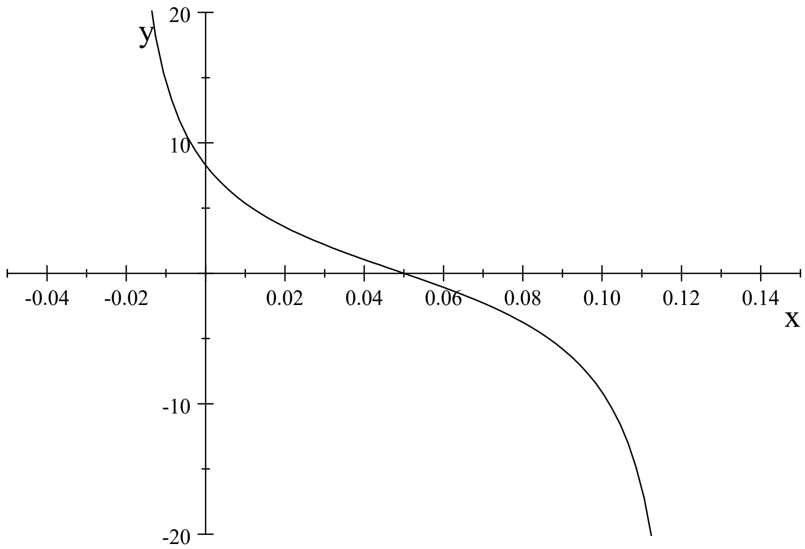

Figure 5: The optimal investment strategy in the Markowitz methodology

It is interesting that the Markowitz approximation has a very similar overall shape as the Thorp binomial approximation. In fact, over the range $0.02 < r < \widehat{\mu}$, the two are practically indistinguishable. In Figure 6, the two are drawn for values of r from about -0.03 up to 0.12. The black curve is λ_M^* and the red curve is λ_B^*.

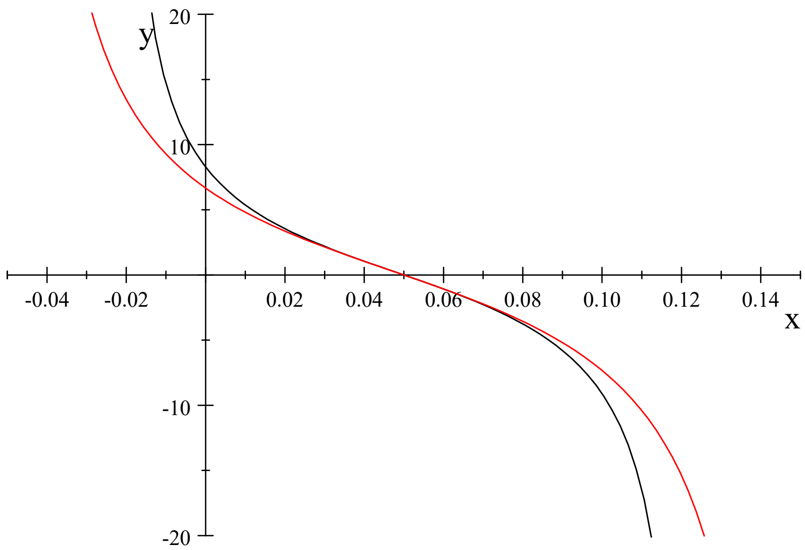

Figure 6: The Thorp approximation, and the Markowitz approximation

Above I looked at Thorp's linear approximation, and the first-order Taylor's approximation to λ_B^*, as more useful constructs for estimating the optimal strategy. It therefore stands to reason that the same should happen with λ_M^*, so I now go on to consider the first-order Taylor's approximation to λ_M^* around the point $r = \widehat{\mu}$, which can then be analysed side by side with $T(r)$ and $R_T(r)$.

$$\lambda_M^* = \left(\frac{1+r}{\widehat{\mu}-r}\right)\left(\frac{1-\sqrt{1-2S^2}}{\sqrt{1-2S^2}}\right) \qquad \frac{\widehat{\mu}-r}{\widehat{\sigma}} \equiv S$$

The first-order Taylor's approximation is defined as

$$R_M(r) = \lambda_M^*(\widehat{\mu}) + \lambda_M^{*\prime}(\widehat{\mu})(r-\widehat{\mu})$$

And since $\lambda_M^*(\widehat{\mu}) = 0$, we have

$$R_M(r) = \lambda_M^{*\prime}(\widehat{\mu})(r-\widehat{\mu})$$

Write λ_M^* as

$$\begin{aligned} \lambda_M^*(r) &= \left(\frac{1+r}{\widehat{\mu}-r}\right)\left(\frac{1-k(r)}{k(r)}\right) \\ &= \frac{(1+r)\,(1-k(r))}{(\widehat{\mu}-r)\,k(r)} \end{aligned}$$

where $k(r) \equiv \sqrt{1-2S(r)^2}$, and of course (as defined above) $S(r) \equiv \frac{\widehat{\mu}-r}{\widehat{\sigma}}$. We have

$$\begin{aligned} k'(r) &= \frac{1}{2}\left(1-2S(r)^2\right)^{-\frac{1}{2}}\left(-4S(r)S'(r)\right) \\ &= \frac{-2}{k(r)}\left(-\frac{S(r)}{\widehat{\sigma}}\right) \\ &= \frac{2S}{\widehat{\sigma}k} \end{aligned}$$

and

$$\begin{aligned} k''(r) &= \frac{2S'\widehat{\sigma}k - 2S\widehat{\sigma}k'}{\widehat{\sigma}^2k^2} \\ &= \frac{2\left(S'\widehat{\sigma}k - S\widehat{\sigma}k'\right)}{\widehat{\sigma}^2k^2} \\ &= \frac{2\left(-\frac{1}{\widehat{\sigma}}\widehat{\sigma}k - S\widehat{\sigma}k'\right)}{\widehat{\sigma}^2k^2} \\ &= \frac{-2\left(k + S\widehat{\sigma}k'\right)}{\widehat{\sigma}^2k^2} \end{aligned}$$

Notice that at $r = \widehat{\mu}$, we have $k = 1$, $k' = 0$, and $k'' = \frac{-2}{\widehat{\sigma}^2}$.

Given those observations on k, we have

$$\begin{aligned}
\lambda_M^{*\prime}(r) &= \frac{\left((1-k)-(1+r)\,k'\right)(\widehat{\mu}-r)\,k-(1+r)\,(1-k)\left(-k+(\widehat{\mu}-r)\,k'\right)}{(\widehat{\mu}-r)^2\,k^2} \\
&= \frac{(1-k)\,(\widehat{\mu}-r)\,k-(1+r)\,k'\,(\widehat{\mu}-r)\,k+(1+r)\,(1-k)\,k-(1+r)\,(1-k)\,(\widehat{\mu}-r)\,k'}{(\widehat{\mu}-r)^2\,k^2} \\
&= \frac{(1-k)\,k\left((\widehat{\mu}-r)+(1+r)\right)-(1+r)\,k'\,(\widehat{\mu}-r)\,(k+1-k)}{(\widehat{\mu}-r)^2\,k^2} \\
&= \frac{(1-k)\,k\,(\widehat{\mu}+1)-(1+r)\,k'\,(\widehat{\mu}-r)}{(\widehat{\mu}-r)^2\,k^2}
\end{aligned}$$

Clearly, at $r=\widehat{\mu}$, both the numerator (since at $r=\widehat{\mu}$ we have $k=1$) and the denominator go to 0. Hence, we need to use L'Hopital's rule. Define the numerator as

$$N(r) \equiv (1-k)\,k\,(\widehat{\mu}+1)-(1+r)\,k'\,(\widehat{\mu}-r)$$

and the denominator as

$$D(r) \equiv (\widehat{\mu}-r)^2\,k^2$$

We then have

$$\begin{aligned}
N' &= (1-k)\,(\widehat{\mu}+1)\,k'-k'k\,(\widehat{\mu}+1)-k'\,(\widehat{\mu}-r)+(1+r)\,k'-(1+r)\,(\widehat{\mu}-r)\,k'' \\
&= \left((1-k)\,(\widehat{\mu}+1)-k\,(\widehat{\mu}+1)-(\widehat{\mu}-r)+(1+r)\right)k'-(1+r)\,(\widehat{\mu}-r)\,k'' \\
&= \left((1-2k)\,(\widehat{\mu}+1)-(\widehat{\mu}-r)+(1+r)\right)k'-(1+r)\,(\widehat{\mu}-r)\,k''
\end{aligned}$$

At $r=\widehat{\mu}$, this is 0 (since $k'=0$).

Next, consider the derivative of the denominator:

$$D'(r) = -2\,(\widehat{\mu}-r)\,k^2+2\,(\widehat{\mu}-r)^2\,kk'$$

Again, this is 0 at $r=\widehat{\mu}$. So a first iteration of L'Hopital still yields no solution. We need to go to a second iteration:

$$\begin{aligned}
N'' &= \left(-2k'\,(\widehat{\mu}+1)+2\right)k'+\left((1-2k)\,(\widehat{\mu}+1)-(\widehat{\mu}-r)+(1+r)\right)k'' \\
&\quad -(\widehat{\mu}-r)\,k''+(1+r)\,k''-(1+r)\,(\widehat{\mu}-r)\,k''' \\
&= \left(-2k'\,(\widehat{\mu}+1)+2\right)k' \\
&\quad +\left((1-2k)\,(\widehat{\mu}+1)-2\,(\widehat{\mu}-r)+2\,(1+r)\right)k''-(1+r)\,(\widehat{\mu}-r)\,k'''
\end{aligned}$$

At $r = \widehat{\mu}$ this is

$$\begin{aligned} N''|_{r=\widehat{\mu}} &= ((1-2)(\widehat{\mu}+1) + 2(1+\widehat{\mu}))\, k'' \\ &= \frac{-2(1+\widehat{\mu})}{\widehat{\sigma}^2} \end{aligned}$$

On the other hand

$$D''(r) = 2k^2 - 4(\widehat{\mu}-r)\, kk' - 4(\widehat{\mu}-r)\, kk' + 2(\widehat{\mu}-r)^2 k'k' + 2(\widehat{\mu}-r)^2 kk''$$

At $r = \widehat{\mu}$ this is just

$$D''|_{r=\widehat{\mu}} = 2$$

Combining these two results, we end up with

$$\lim_{r\to\widehat{\mu}} \lambda_M^{*\prime}(r) = \frac{\left(\frac{-2(1+\widehat{\mu})}{\widehat{\sigma}^2}\right)}{2} = \frac{-(1+\widehat{\mu})}{\widehat{\sigma}^2}$$

We can now, finally, construct the first-order Taylor's expansion to $\lambda_M^*(r)$ taken at the point $r = \widehat{\mu}$:

$$R_M(r) = \frac{(1+\widehat{\mu})(\widehat{\mu}-r)}{\widehat{\sigma}^2}$$

This is exactly the same as the linear Thorp approximation that we saw above, that is, $R_M(r) = R_B(r)$. This leads to:

Proposition 4 *The refined linear approximation* $\frac{(1+\widehat{\mu})(\widehat{\mu}-r)}{\widehat{\sigma}^2}$ *is a robust expression for the optimal strategy for growing wealth over time with discrete time portfolio adjustment and without need for an assumption of binomial risk.*

4 Relationship with original Thorp approximation

One final point is in order. How does the discrete time optimal strategy change as the natural time interval gets shorter? This question is important since it shows with much more clarity exactly what is the relationship between the discrete time strategy and the continuous time strategy found originally by Thorp. In his original paper, Thorp 2006 approached continuous time with the following argument; "... subdivide the time interval into n equal independent steps, keeping the same drift and the same total variance. Thus m, s^2 and r are replaced by m/n, s^2/n and r/n, respectively." Here, m is the mean of the risky asset (which in the present paper is denoted by $\widehat{\mu}$), and s is the risky asset standard deviation (in the present paper, $\widehat{\sigma}$). Thorp's continuous time model is then found by considering $n \to \infty$. We can easily replicate this in the models above. Since both methodologies come to the same discrete time optimal strategy, which we can indicate by $\lambda^*(1)$, to recognise that the interval has a single step, we can write the discrete optimal strategy as

$$\lambda^*(1) = \frac{(1+\widehat{\mu})\,(\widehat{\mu}-r)}{\widehat{\sigma}^2}$$

We can then use Thorp's suggestion to subdivide into n equal independent steps, and get

$$\lambda^*(n) = \frac{\left(1+\frac{\widehat{\mu}}{n}\right)\left(\frac{\widehat{\mu}}{n}-\frac{r}{n}\right)}{\left(\frac{\widehat{\sigma}^2}{n}\right)} = \left(1+\frac{\widehat{\mu}}{n}\right)\frac{\left(\frac{\widehat{\mu}}{n}-\frac{r}{n}\right)}{\left(\frac{\widehat{\sigma}^2}{n}\right)} = \left(1+\frac{\widehat{\mu}}{n}\right)\frac{(\widehat{\mu}-r)}{\widehat{\sigma}^2}$$

Clearly, then, we get

$$\lim_{n\to\infty} \lambda^*(n) = \frac{(\widehat{\mu}-r)}{\widehat{\sigma}^2}$$

So, we find Thorp's continuous time optimal strategy is recovered as the limit of the discrete time strategy formula.

5 Conclusions

In this paper I have reconsidered the age-old question of how an investor should optimally share wealth over one risky and one risk-free asset, but under the assumptions that (1) the investor's objective is to maximise the intertemporal growth rate of their wealth, (2) that the investor can make portfolio adjustments only in distrete time, and (3) that the risky asset is defined in each period according to its mean return and the standard deviation thereof. Two methodologies for finding the solution to that problem are considered. The first is the standard approximation provided by Thorp, but that relies upon two important assumptions, namely that the investor is indifferent between any two assets that have the same mean and standard deviation, and second, that the investor works in continuous time. Removing the second of these two assumptions gives us a first approximation that at least conforms with the three main assumptions mentioned above. A further refinement using a first-order Taylor's approximation delivers a more user-friendly approximation that resembles, but that is not equal to, Thorp's continuous time optimal strategy. It is shown that the new approximation better approximates the optimal strategy than does Thorp's one, and that Thorp's continuous time strategy can be easily recovered as the limit of the discrete strategy as the period interval falls to 0.

The second model that I have considered is grounded exactly upon the standard Markowitz mean-standard deviation model, and the approximation sets in directly as a particular utility function, and indifference curve, in that space. Taking the optimal solution along the market opportunities line gives an expression for the optimal fraction of wealth to be invested in the risky asset directly as a function of the mean and standard deviation of the risky asset. The resulting optimal strategy is similar to, but not the same as, that which is derived from Thorp's methodology (without the assumption of continuous time), with the similarity decreasing as one moves to means that are further from the risk-free return. Interestingly, taking the first-order Taylor's approximation to the optimal strategy delivers exactly the same approximate strategy as was found in the linear approximation to Thorp's approximation. Thus, the approximate strategy that is found as the linear approximation in either methodologies appears to be rather robust, and it is more accurate than is simply using Thorp's continuous time model when in fact time is discrete.

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Appendix 1

In each period, the risky interest rate turns out to be either a or b, so after N periods, n of which happened to be at the interest rate a and $N-n$ were at the rate b, where $0 \leq n \leq N$, the investor's wealth would have grown to be worth

$$w_N = (1+(1-\lambda)r+\lambda a)^n (1+(1-\lambda)r+\lambda b)^{N-n}$$

Now, imagine that the investor acts with the specific objective of maximising the growth of terminal wealth over an entire lifetime of investing. The average growth rate of the fund over N periods is given by the $g(N)$ such that

$$\begin{aligned} 1+g(N) &= \left((1+(1-\lambda)r+\lambda a)^n (1+(1-\lambda)r+\lambda b)^{N-n}\right)^{\frac{1}{N}} \\ &= (1+(1-\lambda)r+\lambda a)^{\frac{n}{N}} (1+(1-\lambda)r+\lambda b)^{\frac{N-n}{N}} \end{aligned}$$

Take the limit of this as $N \to \infty$:

$$\begin{aligned} \lim_{N\to\infty}(1+g(N)) &= \lim_{N\to\infty}(1+(1-\lambda)r+\lambda a)^{\frac{n}{N}} (1+(1-\lambda)r+\lambda b)^{\frac{N-n}{N}} \\ &= (1+(1-\lambda)r+\lambda a)^{\lim_{N\to\infty}\frac{n}{N}} (1+(1-\lambda)r+\lambda b)^{\lim_{N\to\infty}\frac{N-n}{N}} \end{aligned}$$

But, from the Law of Large Numbers and the binomial distribution,

$$\lim_{N\to\infty}\frac{n}{N} = p \quad \text{and} \quad \lim_{N\to\infty}\frac{N-n}{N} = 1-p$$

Therefore,

$$\lim_{N\to\infty}(1+g(N)) = (1+(1-\lambda)r+\lambda a)^p (1+(1-\lambda)r+\lambda b)^{1-p}$$

An investor wanting to maximise $g(N)$ will achieve this same outcome by maximising $(1+(1-\lambda)r+\lambda a)^p(1+(1-\lambda)r+\lambda b)^{1-p}$. As before, taking logs and then elevating we can write this objective as

$$e^{p\ln(1+(1-\lambda)r+\lambda a)+(1-p)\ln(1+(1-\lambda)r+\lambda b)}$$

So in fact, the investor wants to invest a constant fraction λ of each periodic wealth such that the periodic expected utility of a hypothetical investor with log utility, $p\ln(1+(1-\lambda)r+\lambda a)+(1-p)\ln(1+(1-\lambda)r+\lambda b)$, is maximised.

The first-order condition for this maximisation problem is

$$\frac{p(a-r)}{1+(1-\lambda^*)r+\lambda^* a} - \frac{(1-p)(r-b)}{1+(1-\lambda^*)r+\lambda^* b} = 0$$

One can easily check that the second-order condition holds. Then, write the first-order condition as

$$p(a-r)(1+(1-\lambda^*)r+\lambda^* b) = (1-p)(r-b)(1+(1-\lambda^*)r+\lambda^* a)$$

which after a short succession of simple steps reduces down to the required result

$$\lambda^* = \frac{(1+r)(pa+(1-p)b-r)}{(a-r)(r-b)}.$$

Appendix 2

Here I look more in detail as to why, in the Markowitz methodology, a solution will only exist for $1-2S^2>0$, which is

$$\frac{1}{\sqrt{2}} > S$$

The exact representation of the indifference curve in the space (μ_w, σ_w) is found by simply setting a given value for $v(\mu_w, \sigma_w)$, and then solving $v(\mu_w, \sigma_w) = \mu_w - \frac{\sigma_w^2}{2\mu_w}$ for μ_w. To that end, set $v(\mu_w, \sigma_w) = v$;

$$\begin{aligned} v &= \mu_w - \frac{\sigma_w^2}{2\mu_w} \\ 2\mu_w v &= 2\mu_w^2 - \sigma_w^2 \\ 2\mu_w^2 - 2\mu_w v - \sigma_w^2 &= 0 \end{aligned}$$

This is a standard quadratic function of μ_w, the solution of which is given by

$$\begin{aligned} \mu_w &= \frac{2v \pm \sqrt{4v^2 - 4\times 2\times(-\sigma_w^2)}}{4} \\ \mu_w &= \frac{2v \pm \sqrt{4v^2 + 4\times 2\times \sigma_w^2}}{4} \\ \mu_w &= \frac{2v \pm 2\sqrt{v^2 + 2\sigma_w^2}}{4} \\ \mu_w &= \frac{v \pm \sqrt{v^2 + 2\sigma_w^2}}{2} \end{aligned}$$

It is apparent that the $\pm$ can only be $+$, since otherwise this equation would indicate $\mu_w < 0$. Therefore, the equation of an indifference curve in MSD space for this problem can be expressed as

$$\mu_w = \frac{v + \sqrt{v^2 + 2\sigma_w^2}}{2}$$

This is drawn in Figure A1, for the case of $v = 1$. Notice that, at the risk-free point, $\sigma_w^2 = 0$, the value of μ_w for the indifference curve is

$$\mu_w = \frac{v + \sqrt{v^2}}{2} = \frac{2v}{2} = v$$

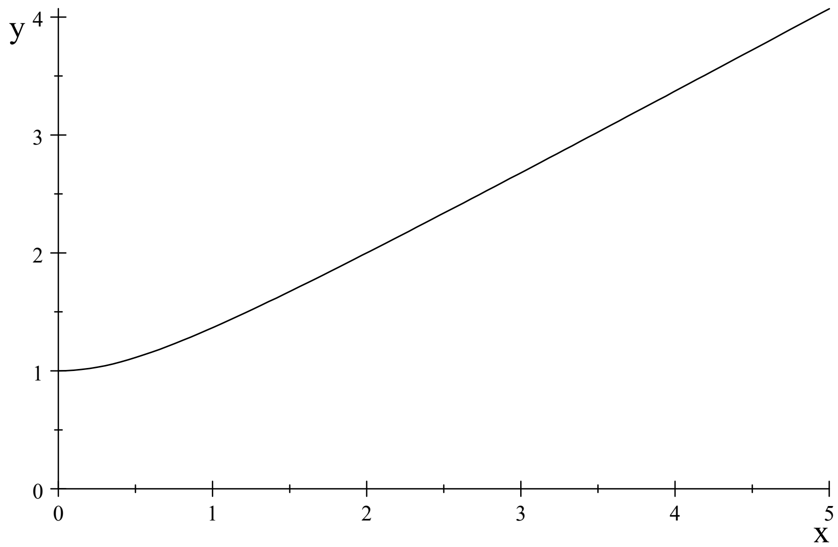

Figure A1: An indifference curve in Markowitz space, drawn for $v = 1$.

Now that we have an equation for the indifference curve, $\mu_w = \frac{v+\sqrt{v^2+2\sigma_w^2}}{2}$, we can easily calculate its slopes:

$$\begin{aligned} \left.\frac{\partial \mu_w}{\partial \sigma_w}\right|_v &= \frac{\frac{1}{2}\left(v^2 + 2\sigma^2\right)^{-\frac{1}{2}} 4\sigma_w}{2} \\ &= \left(v^2 + 2\sigma_w^2\right)^{-\frac{1}{2}} \sigma_w > 0 \end{aligned}$$

and

$$\begin{aligned} \left.\frac{\partial^2 \mu_w}{\partial \sigma_w^2}\right|_v &= -\frac{1}{2}\left(v^2 + 2\sigma_w^2\right)^{-\frac{3}{2}} 4\sigma_w^2 + \left(v^2 + 2\sigma_w^2\right)^{-\frac{1}{2}} \\ &= -2\left(v^2 + 2\sigma_w^2\right)^{-\frac{3}{2}} \sigma_w^2 + \left(v^2 + 2\sigma_w^2\right)^{-\frac{1}{2}} \\ &= \left(v^2 + 2\sigma_w^2\right)^{-\frac{1}{2}} \left(1 - 2\left(v^2 + 2\sigma_w^2\right)^{-1} \sigma_w^2\right) \\ &= \left(v^2 + 2\sigma_w^2\right)^{-\frac{1}{2}} \left(1 - \frac{2\sigma_w^2}{\left(v^2 + 2\sigma_w^2\right)}\right) \end{aligned}$$

Clearly, since $v^2 > 0$, and $\frac{\sigma_w^2}{2(v^2+2\sigma_w^2)} < 1$, we get $\left.\frac{\partial^2 \mu_w}{\partial \sigma_w^2}\right|_v > 0$, that is, the indifference curve is convex.

From the graph, one can see that the indifference curve ends up going very linear as we move upwards along it. In fact, it can be shown that the slope of the curve approaches an asymptote, which is a straight line with positive slope equal to $\frac{1}{\sqrt{2}}$.[13] This is shown in Figure A2, where the asymptote is added (as a red straight line) to the same indifference curve graph as above. The indifference curve approaches the asymptote from above, with a slope that gets ever closer to (but that doesn't ever actually reach) the slope of the asymptote, which is precisely $\frac{1}{\sqrt{2}}$.

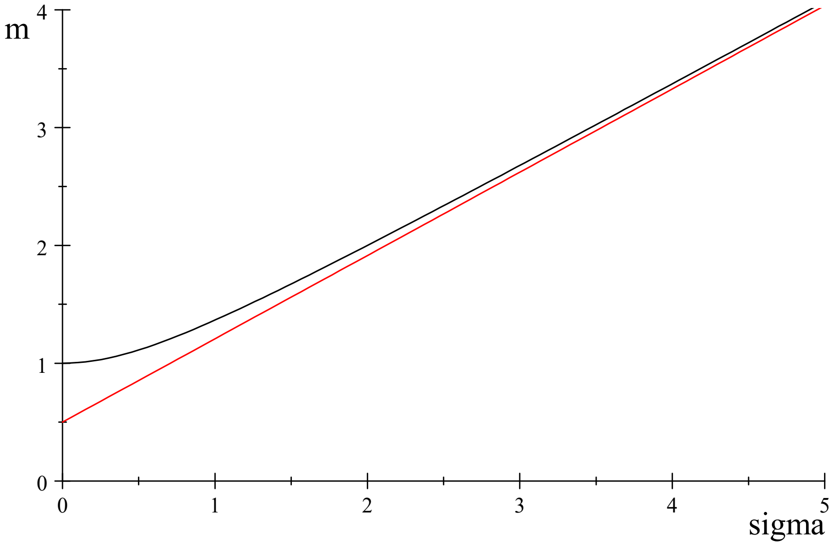

Figure A2: Asymptotic nature of indifference curves in Markowitz space

If we therefore choose a Sharpe ratio S such that the investment opportunities line is steeper than that asymptote, there is no finite solution to the problem – the investor will want to go infinitely to the right.[14] That is why we need to restrict the problem to only have a risky asset with coordinates $(\widehat{\mu}, \widehat{\sigma})$ such that

$$S = \frac{\widehat{\mu} - r}{\widehat{\sigma}} < \frac{1}{\sqrt{2}}.$$

[13] This can be shown mathematically, by calculating $\lim_{x\to\infty} \frac{x}{\sqrt{v^2+2x^2}} = \frac{1}{\sqrt{2}}$.

[14] This is all under the assumption that there is no restriction on short sales. If the investor cannot borrow cash to dedicate to the risky asset, then it is impossible to go beyond the point at which the risky asset is located. In those cases, when $S > \frac{1}{\sqrt{2}}$ all that happens is the investor locates at a corner solution, with all wealth invested in the risky asset (the investor is "all-in").