Modeling VXX

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Abstract

We study the VXX Exchange Traded Note (ETN) that has been actively traded on the New York Stock Exchange in recent years (Whaley, 2013). We confirm the puzzling phenomenon of the significantly negative returns of the VXX that has been reported in the literature. Using the VIX futures pricing framework from Zhang and Zhu (2006) and Zhang, Shu, and Brenner (2010), we create, to our knowledge, the first model of the VXX price which accounts for the fundamental underlying relationships of the SPX (S&P 500 index), the VIX and the VXX. Using our model of the VXX price, we quantify the roll yield and show that the returns of the VXX are driven by the roll yield, as proposed in the literature. The roll yield of any futures position is the return of the position that is not due to movements in the underlying; it is often called the cost of carry in other futures markets. We then show that the sign of the roll yield of the VXX is, on aggregate, driven by the market price of variance risk. We provide a new simple and robust methodology for estimating the market price of variance risk using our model and inputting VXX price. This has to be estimated for any affine model of the SPX which uses a stochastic volatility process, as in the Heston (1993) framework. Our VXX model could be used to price options written on the VXX.

1 Introduction

In this paper, we study the VXX Exchange Traded Note (ETN) that has been actively traded on the New York Stock Exchange in recent years (Whaley, 2013). We re-confirm the puzzling phenomenon of the significantly negative VXX returns. We then create the, to our knowledge, first model of the VXX which encompasses the underlying relationships between the SPX (S&P 500 index), VIX and VXX. Using our model we show that the roll yield is the driver of the VXX's returns and that the roll yield is driven on aggregate by the market price of variance risk. We also provide a new simple way of calculating the market price of variance risk which should simplify the calibration process of affine stochastic volatility models of the SPX, such as in the Heston (1993) framework.

There are three major risk factors which are traded in options markets: market risk, interest rate risk and volatility risk. Market risk is traded in the stock market and interest rate risk is traded in the bond and interest rate derivative markets; however volatility risk has only been traded indirectly in the options market up until recently.

It is well accepted in the literature that both equity returns and the variance of equity returns are random variables (French, Schwert, and Stambaugh, 1987). It is also well understood that the Variance Risk Premium (VRP) is significant and negative (Coval and Shumway (2001) Bakshi and Kapadia (2003) and Carr and Wu (2009)). The VRP is defined as the return of a long position in a variance swap, where you receive the realised variance and pay the variance swap rate. A variance swap is actually a forward contract on the realised variance, but when it was first invented by practitioners it was called a variance swap because returns of the underlying had to be computed daily to obtain the realised variance. The fair variance swap rate is equal to the risk neutral expectation of the realised variance, and therefore the VRP is equal to the realised variance less the risk neutral expectation of the realised variance (Carr and Wu, 2009). The VRP is the negative compensation that investors are willing to pay in order to have a long position in variance. For example, equity investors are willing to take negative returns from a volatility position during normal times, so that when there is a downturn in equity markets and consequently an upturn in volatility, they will receive positive returns from their volatility position, which will offset the losses from their equity position.[1]

Investors trade volatility either to take advantage of the opportunity in the VRP, through a short position in volatility, or to hedge against volatility risk, using a long position in volatility. It is important for investors to manage volatility risk, because volatility changes empart risk on otherwise hedged positions (Demeterfi et al., 1999). Zhu and Zhang (2007) explain that there are two ways that investors can trade volatility. One way investors can take a long position in volatility would be to buy at the money (ATM) options. At the money options will not necessarily stay at the money, and when they are not at the money they have smaller volatility sensitivity (Vega) and therefore become less effective for trading volatility (Zhu and Avellaneda, 1998). When investors trade volatility in the options market their positions are often contaminated by other risk factors, making them inefficient for risk management (Zhu and Zhang (2007) and Demeterfi et al. (1999)). Another way Zhu and Zhang (2007) say investors can trade volatility is through over the counter variance swaps but these are not available to all investors. Developing a financial market to trade volatility directly is therefore very important for researchers and practitioners and has already given investors many more choices for managing volatility risk.

In 1993, the Chicago Board Options Exchange (CBOE) introduced a volatility index, the VIX, using a design by Robert E. Whaley, which was similar to the one proposed by

[1] We use variance and volatility interchangeably, understanding that volatility is the square root of the corresponding annualized variance.

Brenner and Galai (1993). The VIX was computed from the implied volatilities of eight near-the-money S&P 100 index options. The VIX index was the implied 30-day volatility of the S&P 100. In 2003, the methodology for calculating the VIX index changed, and the index using the old methodology was renamed to VXO. The VIX is now calculated using all out-of-the-money (OTM) options on the S&P 500, which have a bid price.As of the 6$^{\text{th}}$ of October 2014, the VIX methodology now uses weekly maturing and monthly maturing SPX options price data, as opposed to just the monthly maturing options price data. Following the change from the old VIX (VXO) to the new VIX, in 2004 the CBOE launched the much anticipated VIX futures and in 2006 the VIX options. These new methods of trading volatility have grown in popularity ever since their inception.

In 2009, S&P Dow Jones Indices started reporting several different VIX futures indices which represent the returns of many different VIX futures positions. One example of a VIX futures index is the S&P 500 VIX Short-Term Futures Total Return Index (SPVXSTR) which tracks the performance of a position in the nearest and second-nearest maturing VIX futures. The SPVXSTR is rebalanced daily to create a constant one-month maturity VIX futures position. Shortly after the VIX futures indices started being reported, Barclays Capital iPath launched the first-ever VIX futures index Exchange Traded Product (ETP), the VXX Exchange Traded Note (ETN). An ETN is unsecured senior debt that pays no coupons (interest) and does not have a fixed redemption at maturity, rather its redemption value is linked to the value of some underlying asset, index or event (Bao, Li, and Gong, 2012). The VXX's redemption value, for example, depends on the value of the SPVXSTR at maturity less an annual management fee of 0.89%.

There are now many different VIX futures ETNs with many underlying indices, all of them combined make the VIX futures ETN market. The VIX futures ETN market has become vastly popular, as all of the ETNs combined have a market capitalization of nearly

4 billion US dollars and average daily dollar trading volume in excess of 800 million US dollars (Whaley, 2013). One of the main drivers of the VIX futures ETN market's growing popularity may be the fact that mutual funds and hedge funds are often restricted from trading futures and options but still have a need to hedge volatility risk, therefore they trade in the VIX futures ETN market instead of trading VIX futures and options directly.

The puzzling phenomenon of the highly negative returns of the VIX futures ETNs (all that have long exposure to the underlying indices) is well documented all throughout the relative literature. We re-document the VXX's returns in table 1 which shows the summary statistics of the VXX, SPX (S&P 500 index ETP) and VIX returns from 30$^{\text{th}}$ January 2009 to the 27$^{\text{th}}$ June 2014. Note the abysmal performance of the VXX, as can be seen firstly by the -0.32% average daily discrete return and the average daily continuously compounded return of -0.39% as opposed to the average daily continuously compounded and discrete returns of the VIX, which were -0.09% and 0.15%, respectively. Secondly, the discrete Holding Period Return (HPR) shows that within our sample period the VXX has lost 99.59% of its value; the VIX has only lost 71.66%. The Compound Annual Growth Rate (CAGR) of -63.49% of the VXX compared with a CAGR of -20.41% of the VIX further displays the underperformance of the VXX. In figure 2, we plot the VIX index, the VXX price and the constant 30-day-to-maturity VIX futures price, as in Zhang, Shu, and Brenner (2010), so the difference is visually observable. We will later show that the main reason why the VXX does not follow the VIX, as the constant 30-day-to-maturity VIX futures contract does, is due to the roll yield.

The VIX futures ETNs have been marketed by issuers and exchanges as great diversification tools due to the asymmetric negative relationship between the VIX and the S&P 500 index. Daigler and Rossi (2006), Moran and Dash (2007) and Chen et al. (2011) all show the diversification benefits of adding the VIX index to an equity portfolio, but the VIX index itself is not investable. Warren (2012) examines the benefits of VIX futures as diversification tools and shows that they could benefit a typical pension portfolio, but his study is ex-post and therefore not conclusive. Deng, McCann, and Wang (2012) show that ETNs on VIX futures indices, such as the VXX, are not very effective hedging/diversification tools for equity and mixed equity and bond portfolios. Alexander and Korovilas (2012) perform the first ex-ante study of VIX futures and their ETNs as diversification tools. They find that in both the Markowitz (1952) and Black and Litterman (1992) frameworks investors would frequently choose to diversify with VIX futures and their ETNs, but that these portfolios would subsequently underperform unless it was during a crisis period. Hancock (2013) tests the performance of VIX futures ETNs as a single investment and when used to diversify a equity position, against three benchmarks, he shows that the VXX and other VIX futures ETNs never consistently outperform benchmarks even when used to diversify equity portfolios. Hancock (2013) suggests that the poor performance is unique to VIX futures ETNs and is not a property of volatility.

Even with the well-documented and easily observed underperformance, the VXX market has made great strides in popularity. Figure 3 shows us the upward trend in the daily dollar trading volume and the initial increase in and then levelling off in market capitalization of the VXX since its inception.

Many articles in the literature suggest that the roll yield of VIX futures is the cause for the underperformance of the VXX. Liu and Dash (2012) quantifies the roll yield and show that on average the roll yield of the SPVXSTR is -0.18% daily. We will use our model for the VXX to quantify its roll yield and investigate the hypothesis that the roll yield drives the significant negative returns of the VXX. Alexander and Korovilas (2013) show that the VXX and other ETNs can be a useful investment when using them for "roll yield arbitrage" where your short the short term ETN, such as the VXX, and long the long term ETN, such as the VXZ, to take advantage of the convexity of the VIX futures term structure

The roll yield is the return that a futures investor captures when their futures contract converges to the spot price as it matures and is not due to changes in the spot price.[2] It is the return that a futures investor captures when rolling a shorter term futures contract into a longer term futures contract. We can define the roll yield of any futures position mathematically as

$$RY_t = \frac{\partial \ln F_t^T}{\partial t} dt, \tag{1}$$

where RY_t is the roll yield at time t, $\ln F_t^T$ is the natural log of the futures price function,T is the time at which the futures contract matures and t is the corresponding time period.[3]

When the market is in backwardation (i.e. downward-sloping term structure of futures prices), the price rolls up to the spot price as maturity approaches; therefore the roll yield will be positive. When the market is in contango (i.e. upward-sloping term structure of futures prices), the price rolls down to the spot price as maturity approaches; therefore the roll yield will be negative. The VIX futures term structure is in contango during normal times, and therefore the roll yield of VIX futures is usually negative. The VIX futures term structure can also be in backwardation, usually during large economic downturns, where the roll yield will become positive and in this case VIX futures ETNs can be profitable (Whaley, 2013). If the VIX futures term structure is in backwardation, we expect that the roll yield will be positive and then the VXX price will actually increase faster than the VIX index, as it will get the positive returns from the increase of the underlying VIX index and the returns from the positive roll yield.

We create a model for VXX price using the VIX futures price approximation from

[2] The spot price refers to the price or level of the underlying asset/index of the futures contract, for example the spot price for a VIX futures contract is the level of the VIX futures index.

[3] This equation for the roll yield holds for a futures contract written on any underlying asset/index.

Zhang, Shu, and Brenner (2010), which we review in section 3.1. We propose the first stochastic model of the VXX which accounts for the underlying dynamics of the S&P 500 index (SPX) and the VIX index. We believe the relationship between the VXX, the VIX and the S&P 500 is essential in building a comprehensive model. We show that the difference between the 30-day-to-maturity VIX futures price change and the VXX price change in figure 2 is in fact due to the roll yield. We show that the typically negative value of the roll yield is driven by the market price of variance risk, λ, which is usually negative.[4]

In the next section we show the methodology for how the SPVXSTR index is calculated. In section 3, we review the theory behind pricing the VIX and VIX futures from Zhang and Zhu (2006) and Zhang, Shu, and Brenner (2010) and use this to create a stochastic model for the VXX price. We then use this model to examine the roll yield of the VXX. In section 4, we use the VXX model to develop a simple way of estimating the market price of variance risk. We then examine the effect of the rebalancing frequency of the SPVXSTR which is also a robustness test of our continuous time VXX model in section 5. Finally in section 6, we conclude and discuss our findings.

2 The SPVXSTR index

To model the VXX, we must first understand the SPVXSTR. In this section, we present our interpretation of the methodology for calculating the SPVXSTR index from S&P Dow Jones Indices (2012).

The SPVXSTR index seeks to model the outcome of holding a long position in short-term VIX futures, specifically positions in the nearest and second-nearest maturing VIX futures contracts. The position is rebalanced daily to create a constant rolling one-month

[4]The market price of variance risk is not to be confused with the variance risk premium mentioned on page 1, we will elaborate on this in Section 3.3.

maturity VIX futures position (Barclays, 2013).

The SPVXSTR index is calculated by

$$SPVXSTR_t = SPVXSTR_{t-1}(1 + CDR_t + TBR_t), \tag{2}$$

where $SPVXSTR_t$ is the index level at time t, $SPVXSTR_{t-1}$ is the index level at time $t-1$, CDR_t is the Contract Daily Return of the VIX futures position and TBR_t is the Treasury Bill Return earned on the notional value of the position. The TBR_t is given by

$$TBR_t = \left[\frac{1}{1 - \frac{91}{360}TBAR_{t-1}}\right]^{\frac{\Delta_t}{91}}, \tag{3}$$

where Δ_t is the number of calendar days between the current and previous business days. $TBAR_{t-1}$ is the Treasury Bill Annual Return, which is equal to the most recent weekly high discount rate for 91-day US Treasury bills effective on the preceding business day. Usually the rates are announced by the US Treasury on each Monday, but if the Monday is a holiday then Fridays rates will apply. The CDR_t is calculated by

$$CDR_t = \frac{w_{1,t-1}F_t^{T_1} + w_{2,t-1}F_t^{T_2}}{w_{1,t-1}F_{t-1}^{T_1} + w_{2,t-1}F_{t-1}^{T_2}} - 1, \tag{4}$$

where the CDR represents the contract daily return, which is the one day discrete return of the underlying VIX futures position of the SPVXSTR, $w_{i,t-1}$ is the weight in the i^{th} nearest maturing VIX futures at time $t-1$, $F_t^{T_i}$ is the market price of the i^{th} nearest maturing VIX futures contract at time t and $F_{t-1}^{T_i}$ is the market price of the i^{th} nearest maturing VIX futures contract at time $t-1$.[5] The weights are rebalanced daily to be

[5] In Equation (4), we use $w_{1,t-1}$ and $w_{2,t-1}$ in the numerator. Deng, McCann, and Wang (2012)CDR_t use $w_{1,t}$ and $w_{2,t}$ which is inconsistent with the methodology from S&P Dow Jones Indices (2012). When calculating discrete returns of any position, the weights should stay constant over the period you are calculating the return for and only the prices should change.

$$w_{1,t} = \frac{dr}{dt},$$

and

$$w_{2,t} = 1 - \frac{dr}{dt},$$

where S&P Dow Jones Indices (2012) defines "dr =The total number of business days within a Roll Period beginning with, and including, the following business day and ending with, but excluding, the following CBOE VIX Futures Settlement Date. The number of business days includes a new holiday introduced intra-month up to the business day preceding such a holiday." and "dt =The total number of business days in the current Roll Period beginning with, and including, the starting CBOE VIX Futures Settlement Date and ending with, but excluding, the following CBOE VIX Futures Settlement Date. The number of business days stays constant in cases of a new holiday introduced intra-month or an unscheduled market closure" (S&P Dow Jones Indices, 2012, p. 7) Figure 1 shows the determination of dr and dt in a diagram for convenience of understanding.

3 Modeling VXX

3.1 Review of VIX and VIX futures model

To model the VXX, we need to start with a model for VIX futures. Zhang and Zhu (2006) and Zhang, Shu, and Brenner (2010) have developed a model for the VIX and VIX futures; for completeness, we review and combine the results from both in this section. The SPX (S&P 500 index) can be modelled by the following diffusion process with a stochastic process of instantaneous volatility as described by Heston (1993):

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB^P_{1,t}, \tag{5}$$

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB^P_{2,t}, \tag{6}$$

where S_t is the SPX, V_t is the instantaneous variance of the SPX, μ is the expected return from investing in the SPX, θ is the physical measure long-run mean level of the instantaneous variance, κ is the physical measure speed of mean reversion of instantaneous variance and σ_V measures the variance of variance. $B^P_{1,t}$ and $B^P_{2,t}$ are two standard Brownian motions that describe the random noise in the SPX return and variance, respectively; they are correlated by a constant correlation coefficient ρ.

The change of probability measure between the physical and risk-neutral parameters are given by

$$\theta = \frac{\theta^* \kappa^*}{\kappa} \tag{7}$$

and

$$\kappa^* = \kappa + \lambda, \tag{8}$$

where κ^* is the risk-neutral speed of mean reversion of volatility, θ^* is the risk-neutral long-run mean level of instantaneous variance and λ is the market price of variance risk, i.e. λ is the risk premium required by taking the risk of $dB^P_{2,t}$. Using the change of probability measure from physical to risk neutral measure parameters Zhang and Zhu (2006) describe the risk-neutral dynamics of the SPX as follows:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dB^*_{1,t}, \tag{9}$$

$$dV_t = \kappa^*(\theta^* - V_t)dt + \sigma_V \sqrt{V_t} dB^*_{2,t}, \tag{10}$$

where r is the risk-free rate, and $dB^*_{1,t}$ and $dB^*_{2,t}$ are two new standard Brownian motions which are correlated by the constant correlation coefficient, ρ. The squared VIX is equal to the variance swap rate, which is equivalent to the conditional expectation in the risk-neutral measure (Carr and Wu, 2009). The VIX squared can therefore given by

$$VIX_t^2 = E_t^* \left[\frac{1}{\tau_0} \int_t^{t+\tau_0} V_s ds \right] = (1-B)\theta^* + BV_t, \tag{11}$$

where t is time, $\tau_0 = \frac{30}{365}$ and $B = \frac{1-e^{-\kappa^* \tau_0}}{\kappa^* \tau_0}$ (Zhang and Zhu, 2006). Zhang and Zhu (2006) then solve for the the VIX futures price formula which is given by

$$\begin{aligned} \frac{F_t^T}{100} = E_t^*(VIX_T) &= E_t^* \left(\sqrt{(1-B)\theta^* + BV_T} \right) \\ &= \int_0^{+\infty} \sqrt{(1-B)\theta^* + BV_T} f^*(V_T|V_t) dV_T, \end{aligned} \tag{12}$$

where the transition probability density of V_T as given by Cox et al. (1985) is

$$f^*(V_T|V_t) = ce^{-u-v} \left(\frac{v}{u} \right)^{q/2} I_q(2\sqrt{uv}), \tag{13}$$

where

$$c = \frac{2\kappa^*}{\sigma_V^2 (1 - e^{-\kappa^*(T-t)})}, \quad u = cV_t e^{-\kappa^*(T-t)}, \quad v = cV_T, \quad q = \frac{2\kappa^* \theta^*}{\sigma_V^2} - 1,$$

where $I_q(.)$ is the modified Bessel function of the first kind and of order q. The distribution function is the non-central chi-square, $\chi^2(2v; 2q+2, 2u)$, with $2q+2$ degrees of freedom and parameter of non-centrality $2u$ proportional to V_t (Zhang and Zhu, 2006). Note that $(T-t)$ is the time to maturity, in years, of the VIX futures contract.

Equation (12) is the accurate formula for the VIX futures price from Zhang and Zhu (2006) using our notation. Zhang, Shu, and Brenner (2010) provide us with a very good closed-form approximation of equation (12) given by [6]

$$\frac{F_t^T}{100} = F_0 + F_1 + F_2, \tag{14}$$

where

$$F_0 = [\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t B e^{-\kappa^*(T-t)}]^{\frac{1}{2}},$$

$$\begin{aligned} F_1 = &-\frac{\sigma_V^2}{8}[\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t B e^{-\kappa^*(T-t)}]^{\frac{-3}{2}} \\ &\times B^2 \left[V_t e^{-\kappa^*(T-t)} \frac{1 - e^{-\kappa^*(T-t)}}{\kappa^*} + \theta^* \frac{(1 - e^{-\kappa^*(T-t)})^2}{2\kappa^*} \right], \end{aligned}$$

$$\begin{aligned} F_2 = &\frac{\sigma_V^4}{16}[\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t B e^{-\kappa^*(T-t)}]^{\frac{-5}{2}} \\ &\times B^3 \left[\frac{3}{2} V_t e^{-\kappa^*(T-t)} \frac{(1 - e^{-\kappa^*(T-t)})^2}{\kappa^{*2}} + \frac{1}{2}\theta^* \frac{(1 - e^{-\kappa^*(T-t)})^3}{\kappa^{*2}} \right], \end{aligned}$$

where $F_1 + F_2$ is a convexity adjustment from the Taylor series expansion of equation (12).

3.2 Nearly 30-day VIX futures

Through numerical computation we have obtained the following approximation of the nearly 30-day-to-maturity VIX futures contract in proposition 1 below.

[6] In Zhang, Shu, and Brenner (2010) θ is assumed to be time dependant, θ_t, but we stick with the simpler version of the model from Zhang and Zhu (2006) and assume that θ is constant. This is a special case specification for an arbitrary process of θ_t, as our results are independant of the specification of this process

Proposition 1. *The price of nearly 30-day-to-maturity VIX futures can be given by*

$$\frac{F_t^T}{100} = [\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}]^{\frac{1}{2}}, \tag{15}$$

with some very small error when compared with the accurate VIX futures price formula from Zhang and Zhu (2006). For example, for the range of parameters $\sigma_V = 0.1$ *to* 0.7, $V_t = 0.04$ *to* 0.20, $\kappa^* = 4$ *to* 7, *constant* $\theta^* = 0.1$ [7] *and maturity of 30 days, the Root Mean Squared Error (RMSE) from using equation* (15) *instead of the full accurate formula, equation* (12), *is only* 1.29%.

Proof. The results of the numerical exercise presented in table 2 lead us to proposition 1. Table 2 presents the values of estimated VIX futures prices using the full formula from Zhang and Zhu (2006), the closed-form approximation of the full formula from Zhang, Shu, and Brenner (2010), equation (14) and two simplifications of the closed-form approximation, $F_0 + F_1$ and just F_0. From table 2, we can see that for 30-day VIX futures prices using just F_0 creates a very small error from the accurate formula, equation (12). The table shows that the error from using just F_0 instead of the accurate formula, equation (12), is always within 3% when $\theta^* = 0.1$ and V_t ranges from 0.04 to 0.2, κ^* ranges from 4 to 7 and σ_V ranges from 0.1 to 0.7. There is one outlier when $V_t = 0.04$, $\kappa^* = 4$ and $\sigma_V = 0.7$, but the error is only just outside 3% at 3.20%. The root mean squared error (RMSE) from using the simple approximation, equation (15), when compared to the accurate formula from Zhang and Zhu (2006) is 1.29%, which is very acceptable. □

We then take the natural log of equation (15) to get an expression for the natural log price of nearly 30-day-to-maturity VIX futures given by

[7] θ^* is set to 0.1 because this is larger than the average value of 0.045 estimated by Luo and Zhang (2012) by using data from 2 Jan 1992 to 31 Aug 2009 and the error of the VIX futures price formulae is proportional to θ^* therefore we are allowing for more error than if we used their θ^* estimate.

$$\ln\left(\frac{F_t^T}{100}\right) = \frac{1}{2}\ln[\theta^*(1 - Be^{-\kappa^*(T-t)}) + V_t Be^{-\kappa^*(T-t)}], \tag{16}$$

where $\ln(\frac{F_t^T}{100})$ is the natural log the price of nearly 30-day-to-maturity VIX futures contract.

In figure 4, we can see the theoretical term structure of VIX futures using equation (15), the full approximation of VIX futures prices, equation (14) and only the $F_0 + F_1$ segment of the full approximation. We use parameter estimates of $\theta^* = 0.1$, $\kappa^* = 5$, $\sigma_V = 0.1425$ and $V_t = 0.06$ to create an upward-sloping VIX futures term structure, as can be observed during normal times in the VIX futures market. The difference between the points at $t + 30$ and $t + 29$ is equal to the average one-day roll yield of the a rolling position in the VIX futures contract, at any point in time t. The spot return is zero when the underlying instantaneous variance is constant, which means that any return that can be seen is due to the roll yield of VIX futures. It can be seen in the diagram that as you step through time from $t + 30$ to $t + 29$, the return will be negative; therefore the one-day roll yield will be negative when the term structure is upward sloping.

3.3 Model of Contract Daily Return

Through some analysis we have obtained the model of the CDR_t in proposition 2 below.

Proposition 2. *We can model the contract daily return (CDR_t) of the SPVXSTR as the log return of a 30-day-to-maturity VIX futures position, therefore our model of the CDR is given by:*

$$CDR_t = d\ln F_t^T\Big|_{T=t+\tau_0} = d\ln F_t^{t+\tau_0} + RY_t, \tag{17}$$

where

$$
\begin{aligned}
d\ln F_t^{t+\tau_0} =& \frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*\tau_0}} - \theta^* + V_t\right]^{-1} dV_t \\
& - \frac{1}{4}\left[\frac{\theta^*}{Be^{-\kappa^*\tau_0}} - \theta^* + V_t\right]^{-2} (dV_t)^2
\end{aligned} \tag{18}
$$

$$
RY_t = \frac{1}{2}\left[\frac{\kappa^*(V_t - \theta^*)Be^{-\kappa^*\tau_0}}{\theta^* + (V_t - \theta^*)Be^{-\kappa^*\tau_0}}\right] dt, \tag{19}
$$

where $\tau_0 = 30/365$*,* $d\ln F_t^{t+\tau_0}$ *is the change in the log price of a constant 30-day-to-maturity VIX futures contract, and* RY_t *is the roll yield of the SPVXSTR.*

Proof. We can model the change of nearly 30-day log VIX futures price by taking the Taylor series expansion of our simple log VIX futures price formula, equation (16) and using Ito's lemma; this gives us

$$
d\ln F_t^T = \frac{\partial \ln F_t^T}{\partial V_t} dV_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial V_t^2}(dV_t)^2 + \frac{\partial \ln F_t^T}{\partial t} dt. \tag{20}
$$

where $\frac{\partial \ln F_t^T}{\partial t} dt$ is defined as the roll yield and the rest is equal to the constant 30-day-to-maturity VIX futures price. The roll yield of the SPVXSTR is the return of the underlying VIX futures position due to the maturity of the position changing from 30 days to 29 days, from one rebalancing of the position to just before the next rebalancing.

Next, we substitute the partial derivatives into equation (20) to get an equation for the change in the nearly 30-day-to-maturity log futures price, given by:[8]

$$
\begin{aligned}
d\ln F_t^T = & \frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-1} dV_t \\
& - \frac{1}{4}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-2} (dV_t)^2 \\
& + \frac{1}{2}\left[\frac{\kappa^*(V_t - \theta^*)Be^{-\kappa^*(T-t)}}{\theta^* + (V_t - \theta^*)Be^{-\kappa^*(T-t)}}\right] dt.
\end{aligned} \tag{21}
$$

[8]See the Appendix, section A.

The SPVXSTR index is rebalanced daily to maintain a VIX futures position with one-month maturity; therefore we can model the contract daily return (CDR_t) of the underlying futures position as the log return of a 30-day-to-maturity VIX futures position. Therefore using equation (21) we can get a model for the CDR_t of the SPVXSTR, equation (17).

□

Remark. *Our model of the CDR takes the time step from daily to continuous. If we isolate only the effect of time (the roll yield) of our CDR model and converting it to the discrete time, where $\Delta t = 1$ (one day), we get the methodology of calculating the CDR from section 4, as shown by*

$$\begin{aligned} RY_t = \frac{\partial \ln F_t^T}{\partial t} dt &= \frac{lnF_t^T - lnF_{t-\Delta t}^T}{\Delta_t} \Delta t \\ &= ln \left(\frac{F_t^T}{F_{t-\Delta t}^T} \right) \\ &\approx \frac{F_t^T}{F_{t-\Delta t}^T} - 1. \end{aligned} \tag{22}$$

We could substitute dV_t and $(dV_t)^2$ in proposition 2 by any stochastic process of instantaneous volatility, for example equation (10), using Ito's Lemma. To allow flexibility in modelling V_t with different stochastic processes, we do not substitute a process for the instantaneous variance into equation (20). The results we present will hold for any reasonable choice of process for V_t.

3.4 VXX model

Through further analysis we have developed the model for the VXX in proposition 3 below.

Proposition 3. *We can model the VXX using the CDR_t combined with a risk-free return r;*

$$\begin{aligned} d\ln VXX_t = CDR_t + rdt &= d\ln F_t^T\Big|_{T=t+\tau_0} + rdt \\ &= d\ln F_t^{t+\tau_0} + RY_t + rdt \end{aligned} \tag{23}$$

where RY_t is the one-day roll yield of the VXX going from 30-day maturity to 29-day maturity and r is the risk-free return on the notional value of the futures position.

Proof. The change in the SPVXSTR index, and therefore the VXX, is composed of the return of the futures position, the CDR_t, and a risk-free return on the notional of the futures position, TBR_t. Therefore combining a model for the CDR_t and a risk free return will create a model for the VXX.

□

This model of the log VXX price is, to our knowledge, the first attempt in the literature to model the VXX whilst encompassing the underlying relationships between the SPX, VIX and VXX. The model can be used to derive the market price of variance risk, λ, from VXX returns, as is described in section 4. We could also use this model to price VXX options, which are essentially Asian options on the underlying instantaneous variance, V_t. In the next section, we use our VXX model to quantify the roll yield and show that it drives the VXX's returns.

3.5 VXX roll yield

Whaley (2013), Deng, McCann, and Wang (2012) and Husson and McCann (2011) all suggest the roll yield as the reason the VXX's returns are so negative. They never quantify the roll yield or show its impact on the VXX using any quantitative method, which is what we have done. Figure 2 shows us a comparison between the performance of the VIX, the VXX and a constant 30-day-to-maturity VIX futures contract. It is easily observed that the 30-day-to-maturity VIX futures contract follows the VIX index very closely but the VXX does not. The VXX observes large negative returns when compared to the VIX index and the 30-day-to-maturity VIX futures contract. From equation (23), we know that the difference between the 30-day-to-maturity VIX futures return and the VXX return is equal to the roll yield, therefore the difference between the VXX and the 30-day-to-maturity VIX futures price in figure 2 at any point in time is the cumulated roll yield since its inception, which is usually negative because of the upward sloping term structure of VIX futures.

To examine what drives the roll yield of the VXX, we assume that the instantaneous variance, V_t, is constant at the physical measure long-run mean level of instantaneous variance, θ, to produce the aggregate upward-sloping term structure of the VXX [9]. If V_t is constant then $dV_t = 0$, and therefore $d \ln F_t^{t+\tau_0} = 0$ and equation (23) simplifies to

$$RY_t^* = \frac{1}{2}\left[\frac{\kappa^*(\theta - \theta^*)Be^{-\kappa^*\tau_0}}{\theta^* + (\theta - \theta^*)Be^{-\kappa^*\tau_0}}\right] dt, \tag{24}$$

where $\frac{RY_t^*}{dt}$ is the aggregate roll yield of the VXX. To examine what drives the roll yield to be negative during normal times, we can use the change of probability measure from the physical measure to the risk-neutral measure long-run mean level of instantaneous variance, equation (7), and substituting this into equation (24) we get

$$\frac{RY_t^*}{\Delta t} = \frac{1}{2}\frac{\frac{\lambda\kappa^*}{\kappa}Be^{-\kappa^*\tau_0}}{1 + \frac{\lambda}{\kappa}Be^{-\kappa^*\tau_0}}. \tag{25}$$

As all parameters apart from λ are always positive and $\kappa^* = \kappa + \lambda > 0$ (Zhang, Shu, and Brenner, 2010), from equation (25) we can see that the sign of λ, the market price of variance risk, is the driver of sign of the one-day roll yield of the VXX, on aggregate. We

[9] The instantaneous variance is a mean-reverting process with long-term mean level of θ; therefore substituting θ for V_t gives the same result as taking the expectation of V_t and substituting this in. The original Zhang, Shu, and Brenner (2010) formula works for general V_t process, therefore it is also valid for the special case of $V_t = \theta$

conclude that the usually negative roll yield of the VXX is driven by the usually negative (as shown in table 3) λ.

Our findings are consistent with those from Eraker and Wu (2013), as we find a negative market price of variance risk drives the returns of the VXX to be so negative, through the negative roll yield, and they find a negative VRP as the cause of the negative returns of VIX futures positions and VIX futures ETNs. Our findings are consistent with Eraker and Wu (2013) because we know that the VRP and the market price of variance risk are almost proportional (Zhang and Huang, 2010).

4 The Market Price of Variance Risk, λ

When implementing a stochastic volatility model, such as in the Heston (1993) framework, estimating the market price of variance risk is essential. The market price of variance risk is unobservable in the market and there is no clear consensus on the method of estimation yet. Table 3 shows some different authors recent estimates for λ, the risk-neutral measure of the mean-reverting speed of variance, κ^*, and the sample period used. We can see from table 3 that the estimation of λ can vastly vary.

Through further analysis of our VXX model we have found a new simple way of estimating λ using VXX return data, as show in proposition 4 below.

Proposition 4. *We can use the VXX return and an estimate of κ^* to measure λ. λ can be given by*

$$\lambda = \frac{\bar{\lambda}\kappa^*}{\kappa^* + \bar{\lambda}}, \tag{26}$$

where

$$\bar{\lambda} = \frac{2R_E}{(1 - \frac{2R_E}{\kappa^*})Be^{-\kappa^* \tau_0}}. \tag{27}$$

where R_E is the annualized excess return of the VXX over the sample period, given by

$$R_E = \frac{1}{T} \ln \frac{VXX_T}{VXX_0} - r \tag{28}$$

Proof. We substitute $V_t = \theta$ into equation 18 to isolate the effect of the aggregate roll yield, therefore equation 23 simplifies to

$$d \ln VXX_t = \left[\left(\frac{1}{2} \frac{\kappa^*(\theta - \theta^*)Be^{-\kappa^* \tau_0}}{\theta^* + (\theta - \theta^*)Be^{-\kappa^* \tau_0}} \right) + r \right] dt, \tag{29}$$

where $dV_t = 0$.

We can now take the integral of equation (29) from 0 to T with respect to t and then substituting in the change of probability measure from the physical measure long-run mean level of variance to the risk neutral measure long-run mean level of variance, from equation (7), we get

$$R = \ln \left(\frac{VXX_T}{VXX_0} \right) = \frac{1}{2} \frac{\bar{\lambda}Be^{-\kappa^* \tau_0}}{(1 + \frac{\bar{\lambda}}{\kappa^*}Be^{-\kappa^* \tau_0})} T + rT, \tag{30}$$

where R is the continuously compounded return on the VXX over the sample. VXX_T is the last VXX price and VXX_0 is the starting VXX price, in the sample period, rT is the cumulated risk-free return over the sample and $\bar{\lambda}$ is given by

$$\bar{\lambda} = \frac{\lambda \kappa^*}{\kappa} = \frac{\lambda \kappa *}{\kappa^* - \lambda}. \tag{31}$$

We can use the VXX return and an estimate of κ^* to measure λ by solving equation (31) for λ, which gives us equation (26). Then solving equation (30) for $\bar{\lambda}$ we get equation (27)

□

We use the parameter estimate of $\kappa^* = 5.4642$ from Luo and Zhang (2012), as their estimate of κ^* is the most recent available one in the literature and the closest to our sample period, to demonstrate our new methodology of calculating λ. We then use the VXX prices from inception $VXX_0 = 6693.12$ on 30 Jan 2009 and the VXX price at the end of our sample $VXX_T = 28.86$, on 27 Jun 2014.[10] $T = 5.4082$ in years and rT is the cumulative Treasury bill return over the same time period, $rT = TBR_{0,T} = 0.558\%$ as defined in equation (3) from section 2 but cumulated over the entire sample. The cumulated TBR is very small, but this is expected as Treasury bill rates have been almost zero since the recent financial crisis. We input these parameter estimates into equation (26) and (27) from proposition 4 to estimate $\lambda = -6.0211$, with very little need for computing power. This estimate coincides with other authors, as it is negative and of similar magnitude; refer to table 3 for comparison.

This method for estimating the market price of variance risk, λ, makes the calibration of any Heston (1993) model much simpler, as λ is now a function of VXX price and κ^*.

4.1 The Market Price of Variance Risk and the Variance Risk Premium

In the recent literature, the existence of a volatility risk premium is well documented. Coval and Shumway (2001) use classical asset pricing theory to study expected option returns. They show that zero beta at-the-money straddles which are long positions in volatility suffer weekly losses on average of about 3%. Bakshi and Kapadia (2003) construct delta hedged portfolios to empirically show that the market VRP is negative. Carr and Wu (2009) calculate the VRPs for many different stock market indices through replicating

[10] VXX price data from NASDAQ website: www.nasdaq.com/symbol/vxx/historical.

variance swaps using options; they find that the VRP on average is negative. Bondarenko (2013) propose a new strategy in replicating discretely sampled realised variance. They empirically study the price of the variance contract using SPX options from January 1990 to December 2009. They also find a negative VRP which cannot be explained by known risk factors and options returns.

Option pricing models which use a stochastic volatility process use a calibrated parameter called the market price of variance risk; this parameter is used in the change of probability measure between physical and risk neutral measure parameters. Papers which use this concept include Johnson and Shanno (1987) Hull and White (1987), Scott (1987) and Heston (1993). The market price of variance risk is estimated by Lin (2007), Duan and Yeh (2010) and Zhang and Huang (2010) and found to be negative as expected.

Zhang and Huang (2010) show that the market price of variance risk, λ, from the Heston (1993) framework, is almost proportional to the VRP, as defined by Carr and Wu (2009), as long as $\lambda\tau_0$ is small. Their result is shown by

$$VRP = \left[\left(\frac{1}{6}\kappa^*\tau_0 + O({\kappa^*}^2\tau_0)\right)\theta^* + \left(\frac{1}{2} - \frac{1}{3}\kappa^*\tau_0 + O({\kappa^*}^2\tau_0^2)\right)V_t\right]\lambda\tau_0 + O(\lambda^2\tau_0^2), \quad (32)$$

where V_t is the instantaneous volatility of the SPX at time t, κ^* is the risk neutral measure mean reverting speed of the instantaneous volatility of the SPX, θ^* is the long term mean level of the instantaneous volatility of the SPX, $\tau_0 = \frac{30}{365}$, $O(\cdot)$ is a function of order $\lambda^2\tau_0^2$ (Zhang and Huang, 2010). The first part of the equation is obviously proportional to λ, as it is multiplied by $\lambda\tau_0$. The reason the relationship between VRP and λ is almost proportional is the $O(\lambda^2\tau_0^2)$ part of equation (32), which is not proportional to λ but as long as $\lambda\tau_0$ is small (relative to 1), then $\lambda^2\tau_0^2$ will be very small. Therefore, we can consider the market price of variance risk as begin proportional to the instantaneous VRP.

Eraker and Wu (2013) use an economic equilibrium model to show that the abysmal performance of VIX futures and VIX futures index ETPs can be explained by the negative VRP, this is consistent with our finding that the negative market price of variance risk drives the VXX's underperformance, through the roll yield.

5 Rebalancing Frequency of SPVXSTR

In this section, we explore the effect of the rebalancing frequency of the SPVXSTR. We start by replicating the SPVXSTR index using VIX futures prices from the 20$^{\text{th}}$ of December 2005 until the 28$^{\text{th}}$ of March 2014[11], with the methodology from S&P Dow Jones Indices (2012). This replicated SPVXSTR time series is displayed in figure 5, along with the actual SPVXSTR time series over our sample. The actual and replicated indices are almost identical, showing that our replication is accurate.

Figure 6 shows four time series of the replicated SPVXSTR index with different rebalancing frequencies of daily, weekly, bi-weekly and monthly rebalancing. The figure shows that as the rebalancing frequency is decreased from daily to weekly, biweekly and monthly, the SPVXSTR's value decreases. If this effect exists going from daily to more frequent rebalancing, for example hourly, then this would be a problem for our continuous time model. To examine the effect of the rebalancing frequency on the price of the VXX for smaller time steps than daily, we needed a VIX futures price time series that was intraday, but real data for this is only available to us for the last 50 days; therefore we chose to simulate a five-year-long hourly VIX futures price time series.

To simulate the hourly time series of VIX futures prices, we first need a time series of instantaneous volatility, which we get from the physical measure stochastic process of instantaneous volatility, given by

[11] Available at http://cfe.cboe.com/Data/HistoricalData.aspx#VX; accessed on the 20$^{\text{th}}$ of April 2014.

$$dV_t = \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB \tag{33}$$

(Heston, 1993).

We then use the simple VIX futures price approximation, F_0, from equation (14) to find a time series of nearest and second-nearest maturing VIX futures prices. We use $\kappa^* = 5.4642$, as this is the most recent estimation; we use $\lambda = -6.0211$ as calculated in section 2. We propose $\theta = 0.1$ and $\sigma_V = 0.4$ as reasonable value. The results of this section are not sensitive to what parameters are used, as long as they are reasonable. For simplicity, we assume that VIX futures mature every 28 days, that there are no non-trading days, trading hours are 24 hours of the day and that the risk-free rate is zero.

We then use the methodology from section 2 to calculate the SPVXSTR index for five years with different rebalancing frequencies and a starting value of one.

Figure 7 shows the resulting SPVXSTR hourly time series for different rebalancing frequencies from hourly to monthly. We can see in figure 7 that the simulated SPVXSTR time series for hourly and daily rebalancing are almost identical. The rebalancing effect going from daily to hourly rebalancing is therefore very small and not a problem in our model. There is, however, a rebalancing effect if the index is rebalanced less often than daily, this is consistent with our findings using market VIX futures prices. To show that our conclusion on the rebalancing frequency is robust to the term structure of VIX futures, we repeated the above exercise but holding V_t constant at different levels. This allows us to create a time series of SPVXSTR with a upward-sloping (in contango) VIX futures term structure, as shown in figure 8, and a downward-sloping (in backwardation) VIX futures term structure, as shown in figure 9. From figures 8 and 9, we can see that the rebalancing frequency does not significantly impact the SPVXSTR for hourly rebalancing. However, there is a significant effect when going to less frequent rebalancing. In both figures, the

VXX model time series estimated using our model is the continuous limit of the rebalancing time series, as would be expected.

Figures 8 and 9 also show the importance of the roll yield as a driver of the SPVXSTR and subsequently the VXX. The two figures isolate the effect of the term structure on the returns of the SPVXSTR by holding V_t constant, and we know that the roll yield is a result of the term structure of VIX futures. When the term structure is upward sloping, causing a negative roll yield, the simulated level of the SPVXSTR will tend to zero as in figure 8, and when the term structure is downward sloping, causing positive roll yield, the simulated level of the SPVXSTR is exponentially increasing as in figure 9.

6 Conclusions and Discussions

We study the VXX ETP which has been traded very actively on the New York Stock Exchange in recent years. We use the VIX futures price approximation from Zhang, Shu, and Brenner (2010) and simplify it for the nearly 30-day VIX futures contract. From this simplified formula for VIX futures prices, we develop a model for the VXX. Our model is, to our knowledge, the first-ever model of the VXX which encompasses the dynamics of the SPX index and the VIX index. Our model is the simplest way to model the VXX while capturing the relationship between the SPX, VIX and the VXX.

Our model explains the large negative returns of the VXX very well and is in line with the methodology from S&P Dow Jones Indices (2012). Our VXX model allows us to show that the difference in returns of the constant 30-day-to-maturity VIX futures contract, as in Zhang, Shu, and Brenner (2010), and the VXX is due to the roll yield as suggested in the literature. The 30-day maturity VIX futures contract closely tracks the VIX index and therefore we can also conclude that the difference between the VIX index returns and the VXX returns is due to the roll yield. Therefore the greater magnitude of negative returns the VXX compared to the VIX index is due to the roll yield. We then examine the roll yield and show that λ, the market price of variance risk, is the main driver of the VXX's negative roll yield.

We have provided a simple and robust way of measuring λ, using our model and VXX prices. To understand the economic explanation for this, we suggest examining the economic model for VIX ETNs from Eraker and Wu (2013). Their model finds that the negative VRP, which is almost proportional to λ (Zhang and Zhu, 2006), is an equilibrium outcome because a long VIX futures position allows investors to hedge against high-volatility and low-return states, such as exhibited in a financial crisis.

Our continuously rebalanced VXX model is adequate for modelling the daily rebalanced VXX, as the effect of the rebalancing frequency is only significant at less frequent than daily rebalancing.

Our model for the VXX is the first of its kind, as it is the first that includes the relationship between the SPX, the VIX and the VXX, which is fundamental in understanding the VXX. Our model could also be used by practitioners to price options written on the VXX. VXX options can be regarded as Asian options written on the underlying instantaneous variance of the SPX. Bao, Li, and Gong (2012) have created a model for pricing VXX options, but they do not account for the dynamics of the S&P 500 or the VIX, which is essential in modelling the VXX.

Our research shows that the roll yield is the main cause for the negative performance of the VXX, as suggested in the literature. It would be interesting to see whether the roll yield also plays a large part in the returns of other VIX futures ETPs; we expect that it would. Our model could be used with any reasonable stochastic process for the instantaneous variance, V_t, and also a time dependant long-run mean level of variance as in Zhang, Shu, and Brenner (2010) and our results should still hold.

One could use a similar approach to ours to explore the effect of the roll yield on other VIX futures ETNs, but we advise caution in using the simplified VIX futures price formula F_0, as it will be prone to more error at longer maturities. Further research is also needed into the calibration technique best used for our model and its accuracy, although it is theoretically sound. Exploring similar approaches to the one in this paper to create models of other VIX futures ETPs could help further develop the literature around these popular yet mysterious investment products.

Appendix

A. Solving for CDR_t model

From equation (16) and Ito's lemma we get

$$d\ln F_t^T = \frac{\partial \ln F_t^T}{\partial V_t} dV_t + \frac{1}{2}\frac{\partial^2 \ln F_t^T}{\partial V_t^2}(dV_t)^2 + \frac{\partial \ln F_t^T}{\partial t} dt, \tag{34}$$

therefore we need to find each of the partial derivatives $\frac{\partial \ln F_t^T}{\partial V_t}$, $\frac{\partial^2 \ln F_t^T}{\partial V_t^2}$ and $\frac{\partial \ln F_t^T}{\partial t}$. We can do this by taking the first order partial derivatives of equation (15) with respect to V_t and t and the second order partial deriviative with respect to V_t. The partial derivatives are given by

$$\begin{aligned}\frac{\partial \ln F_t^T}{\partial V_t} &= \frac{1}{2}\frac{Be^{-\kappa(T-t)}}{\left[\theta^*(1 - Be^{-\kappa(T-t)}) + V_t Be^{-\kappa(T-t)}\right]} \\ &= \frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-1}\end{aligned}, \tag{35}$$

$$\frac{\partial^2 \ln F_t^T}{\partial V_t^2} = -\frac{1}{2}\left[\frac{\theta^*}{Be^{-\kappa^*(T-t)}} - \theta^* + V_t\right]^{-2} \tag{36}$$

and

$$\frac{\partial \ln F_t^T}{\partial t} = \frac{1}{2}\left[\frac{\kappa^*(V_t - \theta^*)Be^{-\kappa^*(T-t)}}{\theta^* + (V_t - \theta^*)Be^{-\kappa^*(T-t)}}\right]. \tag{37}$$

We then substitute all the partial derivatives into equation (34) giving us the full function of the log futures return shown in equation (21) from section 3.3.

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Table 1: **Summary statistics of the daily returns for the SPY, VIX and VXX.** This table shows the summary statistics and correlations of the VXX, SPX (S&P 500 index ETP) and the VIX index returns from the 2nd February 2009 to the 13th August 2014. R_D represents estimates using discrete daily returns and R_C represents estimates using continuously compounded daily returns. The annualised standard deviation is calculated by multiplying the standard deviation by $\sqrt{252}$. The Holding Period Return (HPR) is the return from the first day to the last day of the sample. The Compound Annual Growth Rate (CAGR) is the constant yearly growth rate that would lead to the corresponding HPR, it is calculated by $CAGR = (HPR + 1)^{\frac{1}{T}} - 1$, where T is the length of the sample in years.

	SPX		VIX		VXX	
	R_D	R_C	R_D	R_C	R_D	R_C
Mean	0.08%	0.07%	0.15%	−0.09%	−0.32%	−0.39%
significance p-value	(0.0121)	(0.0217)	(0.4323)	(0.6226)	(0.0020)	(0.0001)
Standard Deviation (σ)	1.13%	1.13%	7.10%	6.86%	3.81%	3.78%
Annualised σ	18.00%	18.00%	112.70%	108.97%	60.55%	60.06%
Skew	−0.1280	−0.2377	1.3281	0.7659	0.7811	0.5213
significance p-value	(0.0596)	(0.0006)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Excess Kurtosis	4.5210	4.4989	5.6978	3.8815	3.2867	2.8287
significance p-value	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
Holding Period Return	164.48%	97.26%	−71.66%	−126.09%	−99.55%	−540.82%
CAGR	19.25%	–	−20.41%	–	−62.43%	–

Correlations		R_D			R_C	
	SPX	VIX	VXX	SPX	VIX	VXX
SPY	1	−0.7659	−0.7834	1	−0.7710	−0.7846
significance p-value		(0.0000)	(0.0000)		(0.0000)	(0.0000)
VIX	–	1	0.8660	–	1	0.8651
significance p-value			(0.0000)			(0.0000)
VXX	–	–	1	–	–	1

Table 2: **30-day VIX futures price estimation.** This table shows the VIX futures price estimates using four different formulae and range of parameter estimates for V_t, σ_V and κ^*. For this exercise we keep the time to maturity constant at 30 days, $\tau = \tau_0 = \frac{30}{365}$ and θ^* constant at $\theta^* = 0.10$. The first four columns show the hypothetical θ^*, V_t, σ_V and κ^* parameters used in the futures price estimates. The first column of estimated futures prices, labelled by F_0, uses the simple approximation for VIX futures prices, the F_0 part of equation (14). The next column of VIX futures prices, labelled by $F_0 + F_1$, uses the simple formula of VIX futures prices and the first half of the convexity adjustment, $F_0 + F_1$ from equation (14). The $F_0 + F_1 + F_2$ column of VIX futures prices uses the full approximation formula, equation (14), from Zhang, Shu, and Brenner (2010). The final column of VIX futures prices uses the accurate formula, equation (12), from Zhang and Zhu (2006). The columns labelled % error, are the percentage difference of the preceding column of prices from the prices estimated by the accurate formula. The Root Mean Squared Error (RMSE) is calculated for each futures price formula when compared to the accurate VIX futures price formula from Zhang and Zhu (2006).

Parameters				VIX Futures Price estimates						
θ^*	V_t	κ^*	σ_V	F_0	% error	$F_0 + F_1$	% error	$F_0 + F_1 + F_2$	% error	Accurate F
0.1	0.04	4	0.1	25.14	0.07%	25.12	0.00%	25.12	0.00%	25.12
0.1	0.04	4	0.4	25.14	1.08%	24.86	−0.02%	24.89	0.09%	24.87
0.1	0.04	4	0.7	25.14	3.20%	24.30	−0.25%	24.56	0.82%	24.36
0.1	0.04	5.5	0.1	26.32	0.05%	26.31	0.00%	26.31	0.00%	26.31
0.1	0.04	5.5	0.4	26.32	0.77%	26.12	−0.02%	26.13	0.05%	26.12
0.1	0.04	5.5	0.7	26.32	2.27%	25.69	−0.17%	25.85	0.42%	25.74
0.1	0.04	7	0.1	27.26	0.04%	27.25	0.00%	27.25	0.00%	27.25
0.1	0.04	7	0.4	27.26	0.57%	27.11	−0.01%	27.12	0.03%	27.11
0.1	0.04	7	0.7	27.26	1.69%	26.78	−0.12%	26.88	0.24%	26.81
0.1	0.12	4	0.1	33.51	0.05%	33.49	0.00%	33.49	0.00%	33.49
0.1	0.12	4	0.4	33.51	0.82%	33.24	0.00%	33.25	0.05%	33.24
0.1	0.12	4	0.7	33.51	2.53%	32.67	−0.02%	32.83	0.46%	32.68
0.1	0.12	5.5	0.1	33.20	0.04%	33.19	0.00%	33.19	0.00%	33.19
0.1	0.12	5.5	0.4	33.20	0.67%	32.98	0.00%	32.99	0.03%	32.98
0.1	0.12	5.5	0.7	33.20	2.04%	32.53	−0.04%	32.64	0.31%	32.54
0.1	0.12	7	0.1	32.95	0.03%	32.94	0.00%	32.94	0.00%	32.94
0.1	0.12	7	0.4	32.95	0.55%	32.77	0.00%	32.78	0.02%	32.77
0.1	0.12	7	0.7	32.95	1.66%	32.39	−0.04%	32.48	0.21%	32.41
0.1	0.2	4	0.1	40.17	0.04%	40.15	0.00%	40.15	0.00%	40.15
0.1	0.2	4	0.4	40.17	0.62%	39.92	0.00%	39.93	0.03%	39.92
0.1	0.2	4	0.7	40.17	1.94%	39.41	0.01%	39.51	0.27%	39.40
0.1	0.2	5.5	0.1	38.88	0.03%	38.87	0.00%	38.87	0.00%	38.87
0.1	0.2	5.5	0.4	38.88	0.55%	38.67	0.00%	38.68	0.02%	38.67
0.1	0.2	5.5	0.7	38.88	1.69%	38.23	0.00%	38.32	0.21%	38.24
0.1	0.2	7	0.1	37.79	0.03%	37.77	0.00%	37.77	0.00%	37.77
0.1	0.2	7	0.4	37.79	0.48%	37.61	0.00%	37.61	0.02%	37.61
0.1	0.2	7	0.7	37.79	1.47%	37.23	−0.01%	37.30	0.16%	37.24
RMSE				-	1.29%	-	0.06%	-	0.23%	-

Table 3: **$\boldsymbol{\lambda}$ and $\boldsymbol{\kappa^*}$ estimates by various authors.** This table shows the estimated value of λ and κ^* from different authors using different sample periods and estimation methods. The "Standard Error (λ)" column shows the standard error of the λ estimates, although these were not available in the published articles in the table we have provided the standard error of our λ estimation.

Author	Data period	κ^*	λ	Standard Error (λ)
Lin (2007)	21 Apr 2004 - 18 Apr 2006	5.3500	-0.3528	-
Duan and Yeh (2010)	2 Jan 2001 - 29 Dec 2006	-1.7956	-7.5697	-
Zhang and Huang (2010)	18 May 2004 - 17 Aug 2007	1.2989	-19.1184	-
Luo and Zhang (2012)	2 Jan 1992 - 31 Aug 2009	5.4642	†	-
Our estimation	30 Jan 2009 - 27 Jun 2014	5.4642	-6.0211‡	0.113612

† Luo and Zhang (2012) do not give the estimate for lambda, but their article is important here as we use their κ^* estimate.
‡ We use the $\kappa^* = 5.4642$ estimate from Luo and Zhang (2012) and assume that it is accurate for our sample period.

Figure 1: **Understanding the SPVXSTR Roll Period.** This diagram shows how *dr* and *dt* are determined for the calculation of the weights in each VIX futures contract of the SPVXSTR. T_i is the settlement date of the i^th nearest maturing VIX futures, which is 30 days before S&P 500 options maturity date (3rd Friday of every month) and is usually on a Wednesday. $T_i - 1$ is the day before i^th nearest maturing VIX futures settlement and the last day of the roll period. On the last day of the roll period the nearest settling VIX futures is eliminated and the second nearest settling VIX futures becomes the nearest. The *dr* and *dt* are the factors used in the calculation of the weights of each of the VIX futures contracts in the SPVXSTR, as shown in section 2. The roll period represents the time during which the weight in the nearest settling VIX futures contract is gradually replaced by a position in the second nearest VIX futures contract. At the end of the roll period all the weight will be in the second nearest VIX futures contract which then becomes the nearest as the previous nearest contract matures, then the next roll period starts, and the process is repeated.

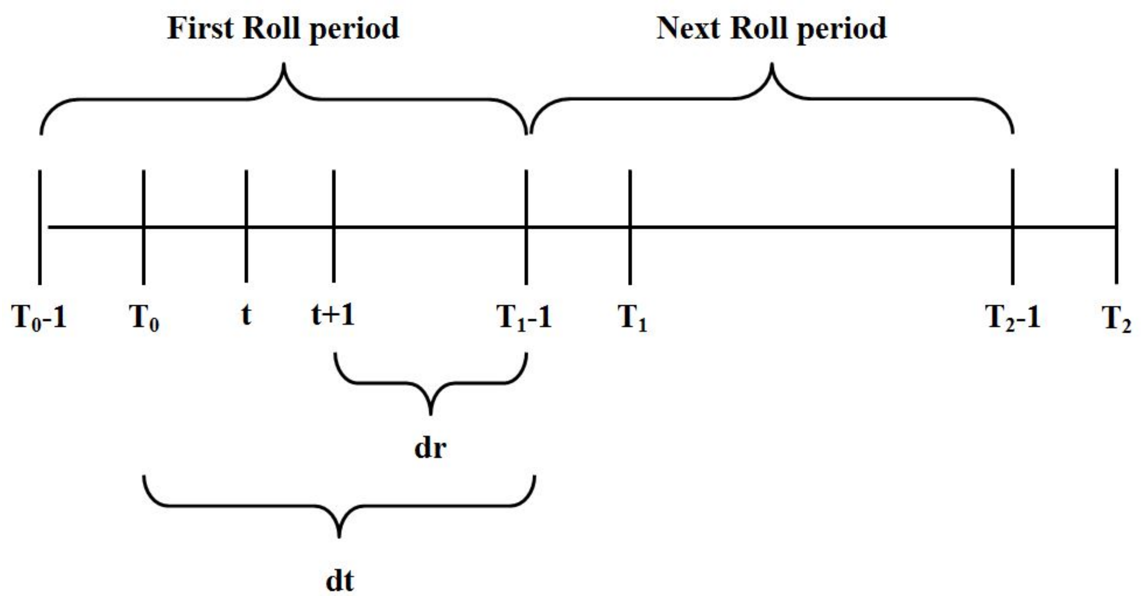

Figure 2: **Historical VIX, 30-day VIX futures price and VXX price.** This figure shows the level of the VIX and the price of 30-day VIX futures on the primary vertical axis and the VXX price on the secondary vertical axis. The 30-day VIX futures contract is the linearly interpolated price of a constant 30-day-to-maturity VIX futures contract, as in Zhang, Shu, and Brenner (2010).

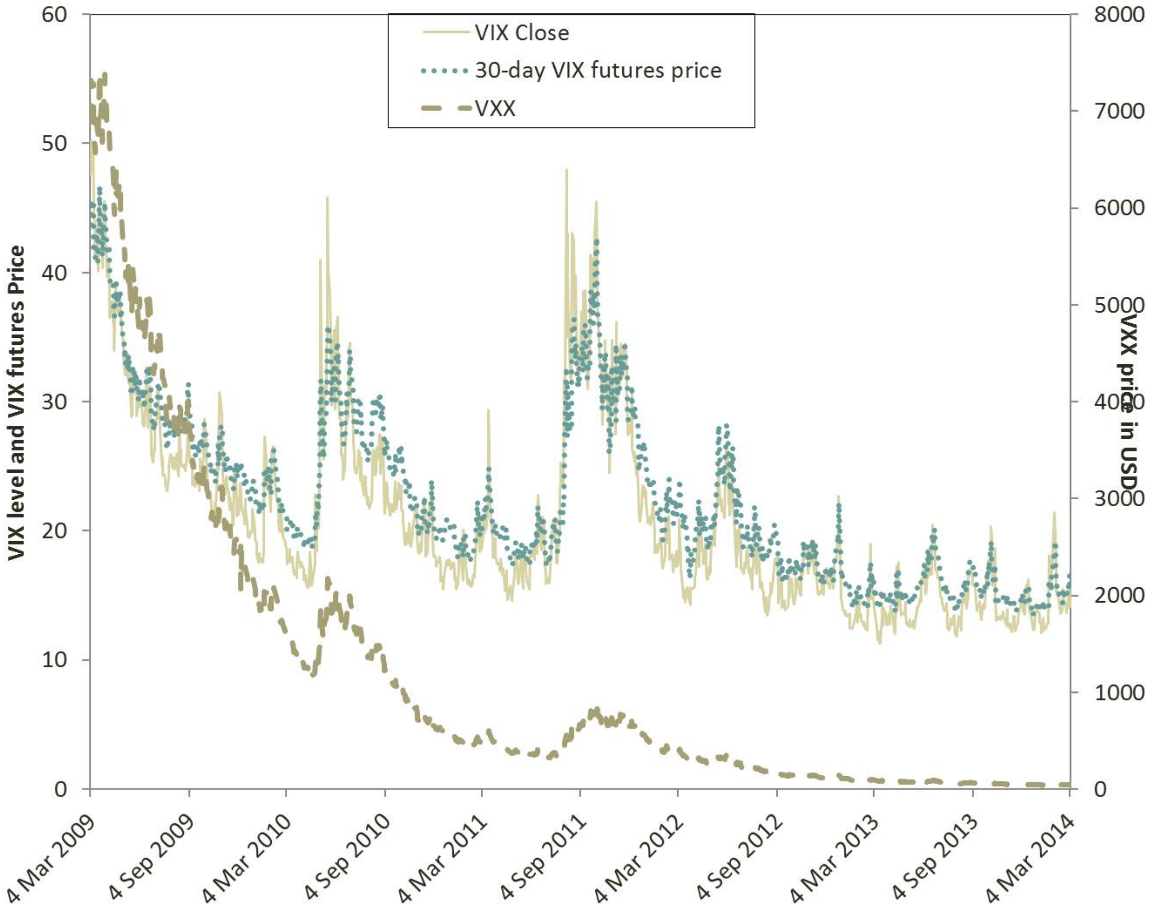

Figure 3: **Market Capitalization and Trading Value of VXX.** This figure shows the daily dollar trading volume and market capitalization of the VXX from the 30$^{\text{th}}$ January 2009 to the 27$^{\text{th}}$ June 2014 in billion US dollars.

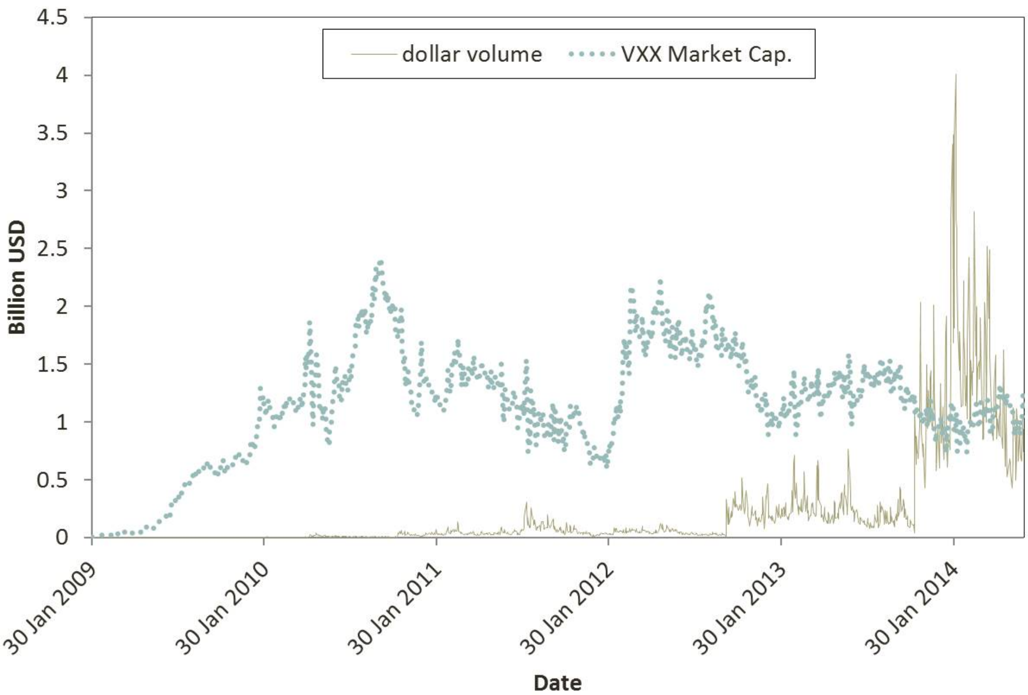

Figure 4: **VIX Term Structure.** This figure shows the term structure of VIX futures prices from 1 day to 50 day maturity calculated using our simple approximation ,F_0, the approximation with the first part of the convexity adjustment, $F_0 + F_1$ and the full approximation from Zhang, Shu, and Brenner (2010) , $F_0 + F_1 + F_2$. These estimated VIX futures prices are calculated using constant parameter estimates of $\theta^* = 0.1$, $\kappa^* = 5$, $\sigma_V = 0.1425$ and $V_t = 0.06$ but the time to maturity varies from 1 day to 50 days.

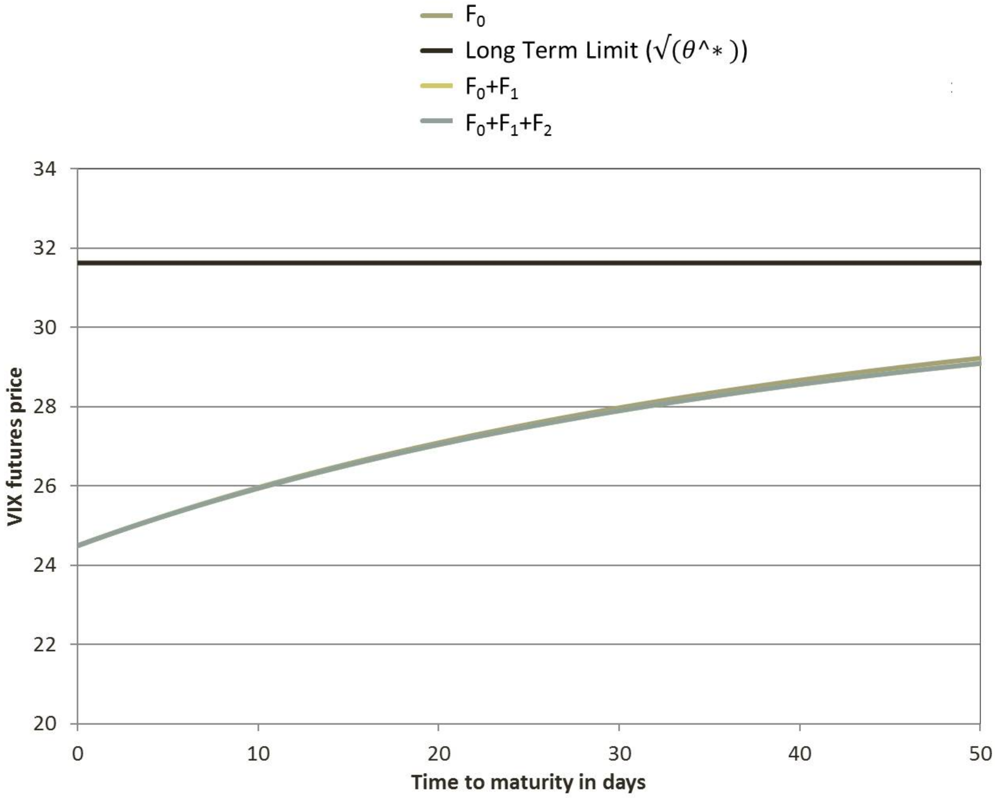

Figure 5: **Replicated vs. Actual SPVXSTR.** This figure shows the actual SPVXSTR time series and our replicated SPVXSTR time series using the methodology from S&P Dow Jones Indices (2012) from the 20th December 2005 until the 28th March 2014.

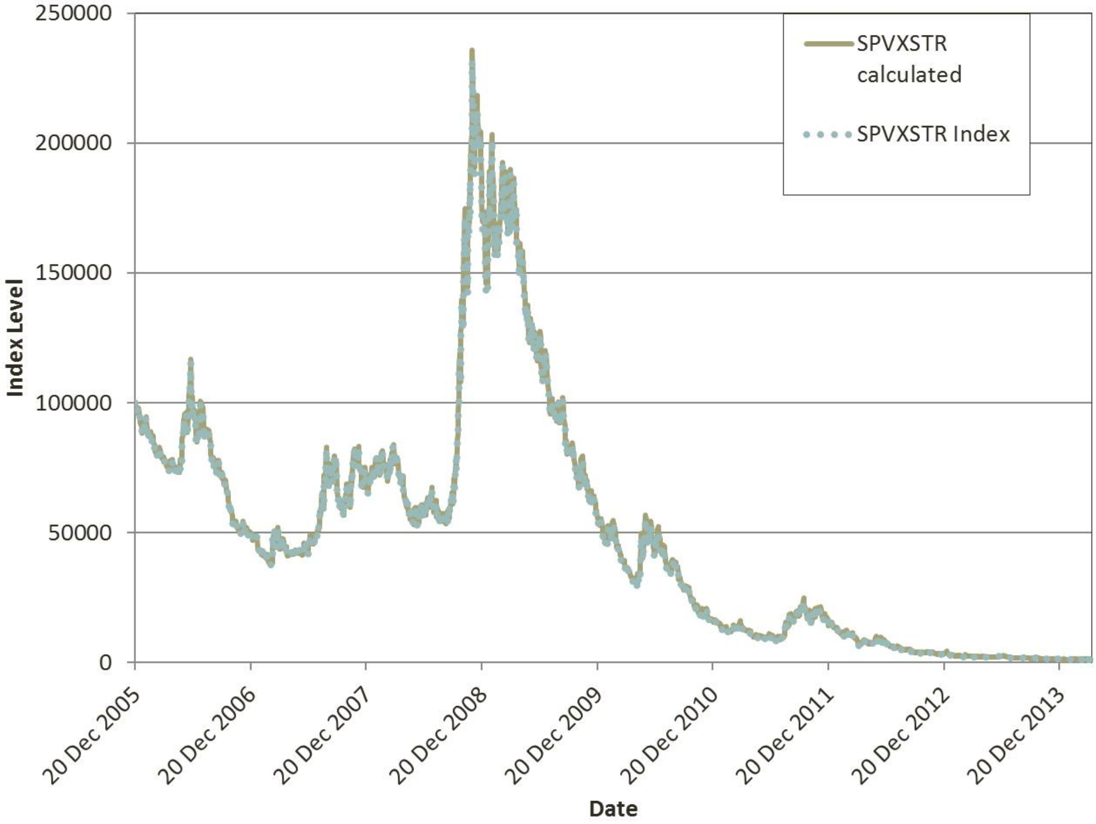

Figure 6: **Replicated SPVXSTR, different Rebalancing frequencies.** This figure shows four different time series of our replication of SPVXSTR. SPVXSTR daily corresponds to daily, SPVXSTR weekly to weekly, SPVXSTR bi-weekly to two weekly and SPVXSTR monthly to monthly rebalancing. The final values of the indices are 1178.63 for daily, 1088.38 for weekly, 842.24 for bi-weekly and 264.60 for monthly rebalancing.

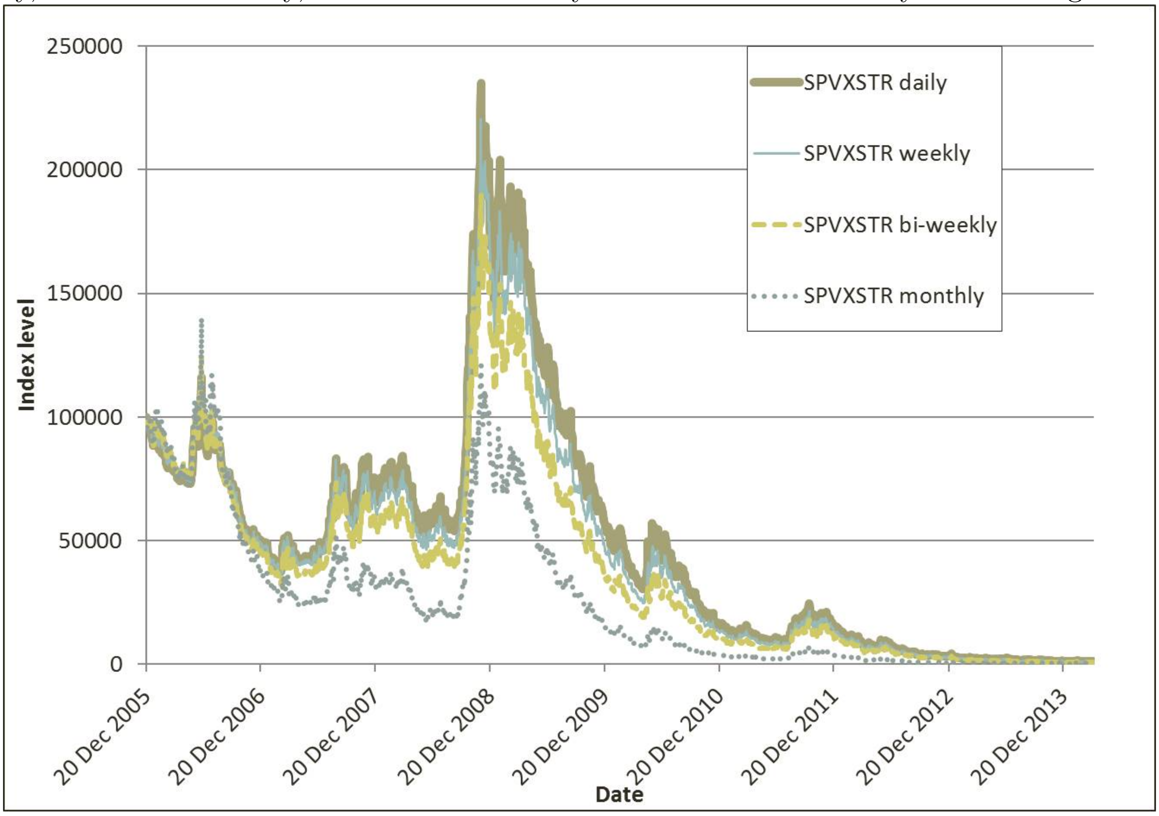

Figure 7: **Simulated index using physical process for V_t.** This figure shows the simulated SPVXSTR index over our 4 year simulation period using $V_0 = 0.02$, $\sigma_V = 0.4$, $\lambda = -6.0211$ the risk-neutral parameter estimates $\kappa^* = 5.4642$ and $\theta^* = 0.1$, the physical process of dV_t as described in equation (33) and the simple VIX futures price formula, F_0, from equation eqrefapproxVIXfuture from Zhang, Shu, and Brenner (2010). The label of each time series corresponds to the rebalancing frequency used.

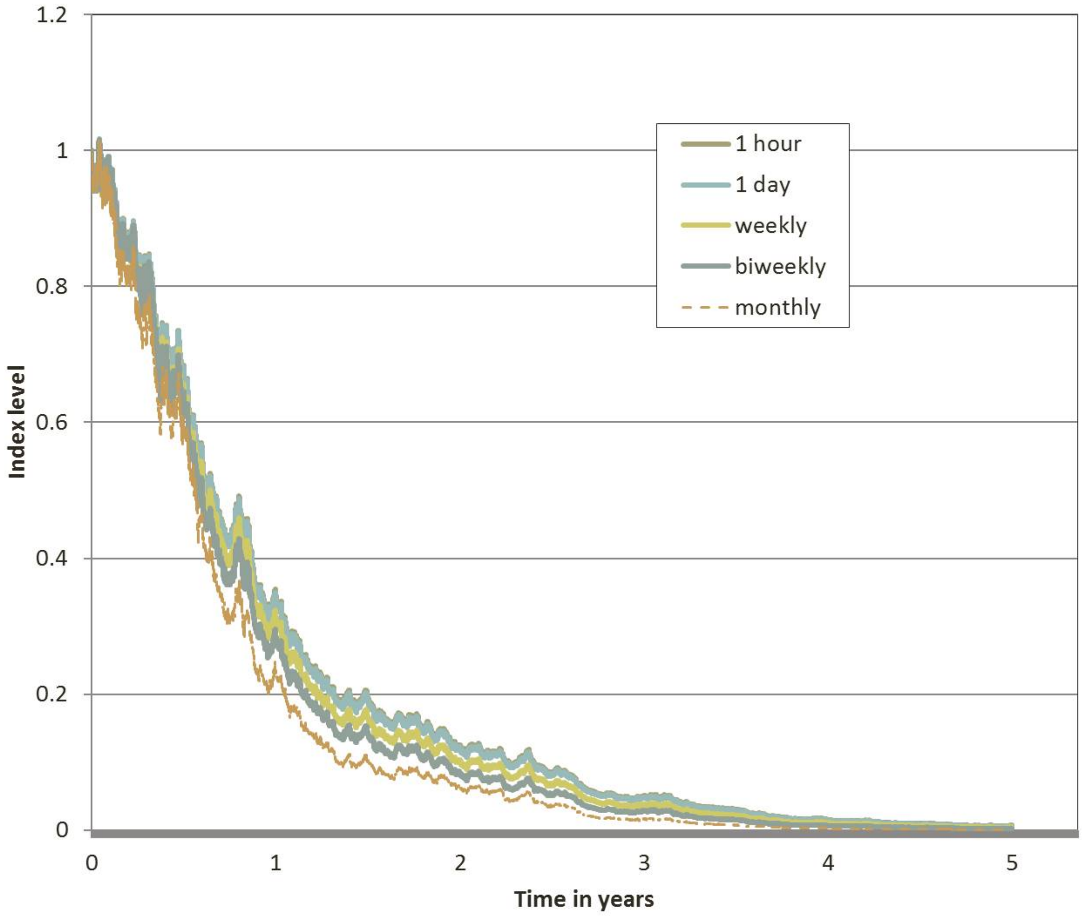

Figure 8: **Simulated index using $\boldsymbol{V_t = \theta < \theta^*}$.** This figure shows the time series of the simulated SPVXSTR, when the instantaneous variance is set constant at $V_t = \theta = 0.0476 < \theta^* = 0.1$ forcing a upward sloping VIX futures term structure. To calculate the futures prices we use the volatility of volatility $\sigma_V = 0.4$ and the risk-neutral parameter estimates $\kappa^* = 5.4642$ and $\theta^* = 0.1$ are used. The label of each time series corresponds to the rebalancing frequency used.

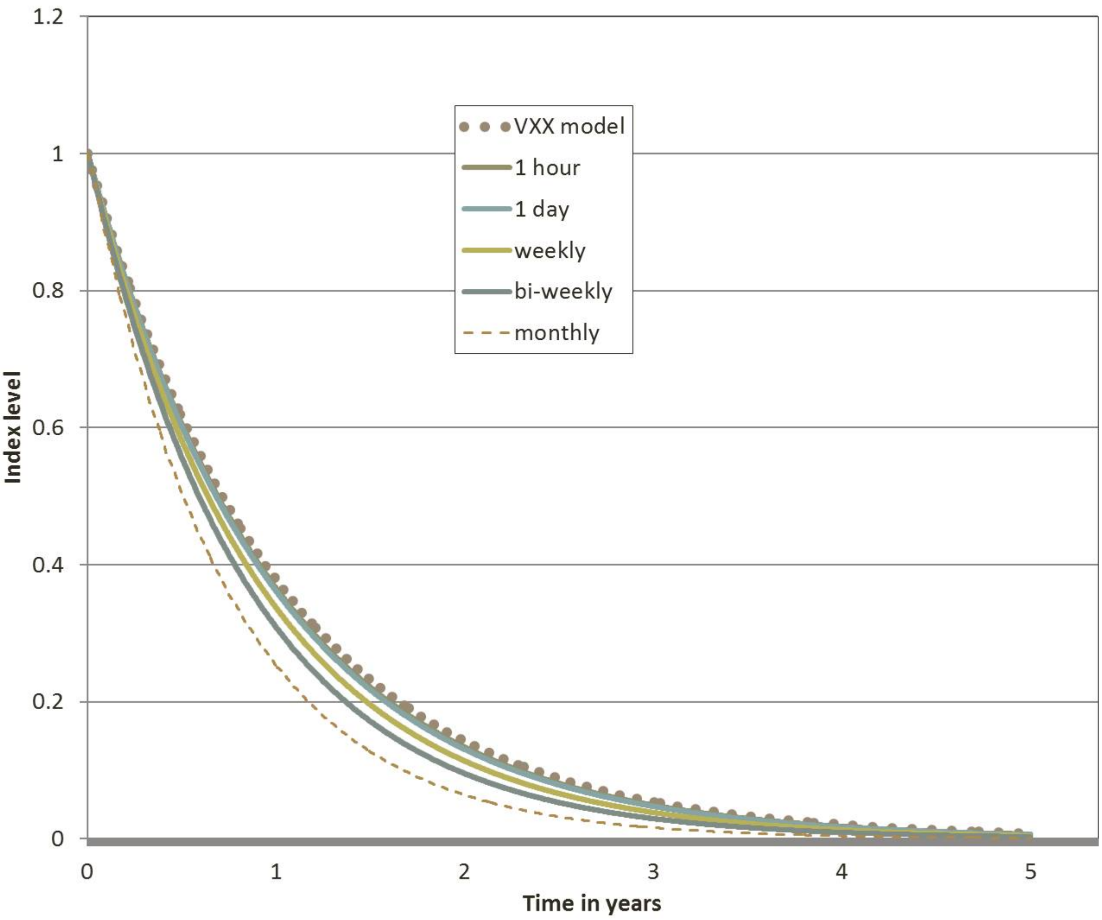

Figure 9: **Simulated index using $V_t = 0.14 > \theta^*$.** This Figure shows the time series of the simulated SPVXSTR, when V_t is set constant at 0.14 which is higher than $\theta^* = 0.1$ forcing a downward sloping VIX futures term structure. To calculate the futures prices we use the volatility of volatility $\sigma_V = 0.4$ and the risk-neutral parameter estimates $\kappa^* = 5.4642$ and $\theta^* = 0.1$ are used. The label of each time series corresponds to the rebalancing frequency used.